Majorization, Doubly Stochastic Matrices, and Comparison of Eigenvalues

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ABSTRACT

1. Basic properties of majorization. 2. Isotone maps and algebraic operations. 3. Double sub- and superstochasticity. 4. Doubly stochastic matrices. 5. Doubly stochastic matrices with minimum permanent. 6. Comparison of eigenvalues. 7. Doubly stochastic maps.

INTRODUCTION

This paper is based on the lectures on majorization I delivered at Hokkaido University and Toyama University in 1981. I have limited myself to the discrete finite-dimensional case in the belief that all the essential aspects of majorization theory can be understood in the discrete setting. The lecture is self-contained except for the Hahn-Banach-type theorem, the Brouwer fixed-point theorem, and the Lagrange multiplier theorem.

The paper is divided into three parts. Part I is classical. The basic properties of majorization are given in Section 1, and the maps that preserve majorization, i.e. isotone maps, are studied in Section 2. In part II structures of doubly (sub- and super-) stochastic matrices are analyzed. The main point of Section 3 is a condition for the existence of a doubly stochastic matrix between two given matrices. In Section 4 the classical Birkhoff theorem is proved together with Sinkhorn's theorem on diagonal equivalence to a doubly stochastic matrix. Section 5 presents a very recent topic: Egorychev's solution of the van der Waerden conjecture on permanents. Part III is devoted to

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comparison of eigenvalues and singular values of matrices in terms of majorization. In Section 6 eigenvalues and singular values of sum and product are studied. Besides, the supremum and infimum of Hermitian matrices are discussed. Section 7 is an elementary introduction to a generalization of the notion of majorization to C^*-algebras.

I. MAJORIZATION

1. *Basic Properties of Majorization*

$\mathbf{C}^n$ is the n-dimensional Hilbert space; for $\vec{x} = (x_1, \dots, x_n)^T$ and $\vec{y} = (y_1, \dots, y_n)^T$

$$\langle \vec{x}, \vec{y} \rangle = \sum_{j=1}^{n} x_j \bar{y}_j \quad \text{and} \quad \|\vec{x}\| = \langle \vec{x}, \vec{x} \rangle^{1/2}. \tag{1.1}$$

Here each $\vec{x}$ is treated as a column vector. The *trace* of $\vec{x}$ is

$$\operatorname{tr}(\vec{x}) := \sum_{j=1}^{n} x_j = \langle \vec{x}, \vec{e} \rangle, \qquad \text{where} \quad \vec{e} = (1, \dots, 1)^T. \tag{1.2}$$

For any real n-vectors $\vec{x}, \vec{y} \in \mathbf{R}^n$ their coordinatewise maximum and minimum are denoted by $\vec{x} \vee \vec{y}$ and $\vec{x} \wedge \vec{y}$ respectively, and

$$\vec{x}^+ := \vec{x} \vee 0 \quad \text{and} \quad |\vec{x}| := \vec{x} \vee (-\vec{x}). \tag{1.3}$$

For any subset $I \subset \{1, \dots, n\}$, denote by $\vec{e}_I$ the n-vector whose jth component is 1 or 0 according as $j \in I$ or $\notin I$. Besides, we use

$$\vec{e}^{(k)} = (\overbrace{1, \dots, 1}^{k}, 0, \dots, 0)^T. \tag{1.4}$$

$|I|$ will denote the number of elements in I.

Given a real vector $\vec{x} = (x_1, \dots, x_n)^T \in \mathbf{R}^n$, let $\vec{x}^{\downarrow} = (x_1^{\downarrow}, \dots, x_n^{\downarrow})^T$ denote its *decreasing rearrangement*, that is,

$$x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$$

are the components of $\vec{x}$ in decreasing order. Similarly $\vec{x}_{\cdot} = (x_{\cdot 1}, \dots, x_{\cdot n})^T$ denotes the *increasing rearrangement* of $\vec{x}$, so that

$$x_{\cdot j} = x^{\cdot}_{n-j+1}, \qquad j = 1, \dots, n. \tag{1.5}$$

$\vec{x}$ is said to be *majorized* by $\vec{y}$ (or $\vec{y}$ majorizes $\vec{x}$), in notation $\vec{x} \prec \vec{y}$, if

$$\sum_{j=1}^{k} x^{\cdot}_j \leqslant \sum_{j=1}^{k} y^{\cdot}_j, \qquad k = 1, \dots, n-1, \tag{1.6}$$

$$\sum_{j=1}^{n} x^{\cdot}_j = \sum_{j=1}^{n} y^{\cdot}_j. \tag{1.7}$$

Since

$$\sum_{j=1}^{k} x^{\cdot}_j = \max_{|I|=k} \langle \vec{x}, \vec{e}_I \rangle, \tag{1.8}$$

an equivalent form of (1.6) is that for any $I \subset \{1, \dots, n\}$ (with $|I| \leqslant n-1$) there exists $J \subset \{1, \dots, n\}$ such that

$$|I| = |J| \quad \text{and} \quad \langle \vec{x}, \vec{e}_I \rangle \leqslant \langle \vec{y}, \vec{e}_J \rangle, \tag{1.6$'$}$$

while (1.7) means

$$\operatorname{tr}(\vec{x}) = \operatorname{tr}(\vec{y}). \tag{1.7$'$}$$

Since by (1.5)

$$\sum_{j=1}^{k} x_{\cdot j} = \sum_{j=1}^{n} x^{\cdot}_j - \sum_{j=1}^{n-k} x^{\cdot}_j,$$

majorization $\vec{x} \prec \vec{y}$ is also expressed by the conditions

$$\sum_{j=1}^{k} x_{\cdot j} \geqslant \sum_{j=1}^{k} y_{\cdot j}, \qquad k = 1, \dots, n-1, \tag{1.9}$$

$$\sum_{j=1}^{n} x_{\cdot j} = \sum_{j=1}^{n} y_{\cdot j}. \tag{1.10}$$

$\vec{x}$ is said to be *submajorized* by $\vec{y}$ (or $\vec{y}$ submajorizes $\vec{x}$), in notation $\vec{x} \underset{\cdot}{\prec} \vec{y}$, if

$$\sum_{j=1}^{k} x_j^{\cdot} \leqslant \sum_{j=1}^{k} y_j^{\cdot}, \qquad k=1,\dots,n. \tag{1.11}$$

Similarly $\vec{x}$ is said to be *supermajorized* by $\vec{y}$ (or $\vec{y}$ supermajorizes $\vec{x}$), in notation $\vec{x} \overset{\cdot}{\prec} \vec{y}$, if

$$\sum_{j=1}^{k} x_{\cdot j} \geqslant \sum_{j=1}^{k} y_{\cdot j}, \qquad k=1,\dots,n. \tag{1.12}$$

Submajorization and supermajorization are stable under multiplication by positive scalars, while multiplication by a negative scalar causes exchange of their roles:

$$\vec{x} \underset{\cdot}{\prec} \vec{y} \quad \text{if and only if} \quad -\vec{x} \overset{\cdot}{\prec} -\vec{y}. \tag{1.13}$$

Since $\vec{x} \prec \vec{y}$ is equivalent to the simultaneous occurrence of $\vec{x} \underset{\cdot}{\prec} \vec{y}$ and $\vec{x} \overset{\cdot}{\prec} \vec{y}$, majorization is stable under multiplication by any real scalar.

Majorization (sub- or supermajorization) introduces a preorder into $\mathbf{R}^n$. Let us write $\vec{x} \sim \vec{y}$ if $\vec{x} \prec \vec{y}$ and $\vec{x} \succ \vec{y}$. Obviously this equivalence relation $\vec{x} \sim \vec{y}$ means that $\vec{x}$ is obtained by permuting the components of $\vec{y}$, or equivalently $\vec{x} = \Pi\vec{y}$ for a permutation matrix Π. Recall that a square matrix Π is a permutation matrix if each row and column has a single unit and all other entries are zero; in other words, there is a permutation π of indices $\{1,\dots,n\}$ such that $\Pi = (\delta_{i\pi_j})$. The order structure introduced by sub- or supermajorization gives rise to the same equivalence relation as $\sim$.

Let us enumerate several examples of natural occurrence of majorization relation.

EXAMPLE 1. Given a vector $\vec{y} \in \mathbf{R}^n$, consider the two sets

$$Y_{\#} := \{\vec{x} : \vec{x} \prec \vec{y}\} \quad \text{and} \quad Y^{\#} := \{\vec{x} : \vec{x} \succ \vec{y}\}.$$

Since $\vec{x} \prec \vec{y}$ implies $y_1^{\cdot} \geqslant x_j \geqslant y_n^{\cdot}$, $j=1,\dots,n$, the set $Y_{\#}$ is topologically bounded, and has a minimum vector with respect to the preorder $\prec$. In fact, $[\mathrm{tr}(\vec{y})/n]\vec{e}$ is the minimum vector. The set $Y^{\#}$ is not, in general, topologi-

cally bounded. But if $\vec{y} \geqslant 0$, then $Y^{\#}$ is bounded from above and has $(\operatorname{tr}(\vec{y}), 0, \dots, 0)^T$ as a maximum vector.

EXAMPLE 2. If $\vec{x} = \vec{x}(\omega)$ is a random n-vector on a probability space, then the average (i.e. mean) vector of $\vec{x}$ is majorized by that of the decreasingly rearranged random vector $\vec{x}^{\cdot}$:

$$E(\vec{x}) \prec E(\vec{x}^{\cdot}).$$

This follows from the identity (1.8):

$$\max_{|I|=k} \langle E(\vec{x}), \vec{e}_I \rangle \leqslant E\Big(\max_{|I|=k} \langle \vec{x}, \vec{e}_I \rangle \Big)$$

$$= E\left(\sum_{j=1}^{k} x_j^{\cdot} \right) = \sum_{j=1}^{k} E(x_j^{\cdot}).$$

EXAMPLE 3. In Example 2 above, let $x_j(\omega) = \chi_{A_j}(\omega)$, $j = 1, \dots, n$, where χ_{A_j} is the characteristic (or indicator) function of an event A_j. Consider the new events defined by

$$B_j := \{\text{at least } j \text{ of } A_1, \dots, A_n \text{ occur}\}.$$

Then it follows that

$$\sum_{j=1}^{k} \chi_{B_j}(\omega) = \max_{|I|=k} \sum_{j \in I} \chi_{A_j}(\omega).$$

Therefore we arrive at the following statement. If a_j is the probability of event A_j, $j = 1, \dots, n$, and if b_j is the probability that at least j of $A_1, \dots, A_n$ occur, then $\vec{a} \prec \vec{b}$.

Let us turn to a characterization of submajorization in terms of a rearrangement-free parameter.

THEOREM 1.1. *$\vec{x} \prec_{\cdot} \vec{y}$ if and only if*

$$\operatorname{tr}(\vec{x} - t\vec{e})^+ \leqslant \operatorname{tr}(\vec{y} - t\vec{e})^+ \qquad \textit{for all} \quad t \in \mathbf{R}. \tag{1.14}$$

Proof. Suppose $\vec{x} \prec \vec{y}$. The inequality (1.14) holds obviously for $t > x_1^{\cdot}$. If $x_k^{\cdot} \geqslant t \geqslant x_{k+1}^{\cdot}$ (with $x_{n+1}^{\cdot} = -\infty$),

$$\operatorname{tr}(\vec{x} - t\vec{e})^{+} = \sum_{j=1}^{k} \left(x_j^{\cdot} - t\right) = \sum_{j=1}^{k} x_j^{\cdot} - kt,$$

while

$$\operatorname{tr}(\vec{y} - t\vec{e})^{+} \geqslant \sum_{j=1}^{k} \left(y_j^{\cdot} - t\right)^{+} \geqslant \sum_{j=1}^{k} y_j^{\cdot} - kt.$$

Now the inequality (1.14) results from definition of $\vec{x} \prec \vec{y}$.

Suppose, conversely, that (1.14) is true. For small t, the inequality (1.14) leads to $\operatorname{tr}(\vec{x}) \leqslant \operatorname{tr}(\vec{y})$. Next, put $t = y_k^{\cdot}$. Then, as above,

$$\operatorname{tr}(\vec{y} - t\vec{e})^{+} = \sum_{j=1}^{k} y_j^{\cdot} - kt,$$

while

$$\operatorname{tr}(\vec{x} - t\vec{e})^{+} \geqslant \sum_{j=1}^{k} x_j^{\cdot} - kt,$$

which yields by (1.14) $\sum_{j=1}^{k} x_j^{\cdot} \leqslant \sum_{j=1}^{k} y_j^{\cdot}$. ■

Corollary 1.2. *$\vec{x} \prec \vec{y}$ if and only if*

$$\operatorname{tr}|\vec{x} - t\vec{e}| \leqslant \operatorname{tr}|\vec{y} - t\vec{e}| \qquad \textit{for all} \quad t \in \mathbf{R}. \tag{1.15}$$

Proof. (1.15) follows from (1.14) and $\operatorname{tr}(\vec{x}) = \operatorname{tr}(\vec{y})$, via a general relation

$$|\vec{z}| + \vec{z} = 2\vec{z}^{+} \qquad \text{for} \quad \vec{z} \in \mathbf{R}^n. \tag{1.16}$$

Conversely the inequality (1.15) for large t yields

$$\operatorname{tr}(t\vec{e} - \vec{x}) \leqslant \operatorname{tr}(t\vec{e} - \vec{y});$$

hence $\operatorname{tr}(\vec{x}) \geqslant \operatorname{tr}(\vec{y})$. Similarly the inequality for small t yields $\operatorname{tr}(\vec{x}) \leqslant \operatorname{tr}(\vec{y})$.

Therefore (1.15) implies $\operatorname{tr}(\vec{x}) = \operatorname{tr}(\vec{y})$. Now (1.14) follows from (1.15) and $\operatorname{tr}(\vec{x}) = \operatorname{tr}(\vec{y})$ via the relation (1.16). Now appeal to Theorem 1.1. ∎

For $\vec{u} \in \mathbf{R}^{n_1}$ and $\vec{v} \in \mathbf{R}^{n_2}$, let us denote by $(\vec{u}, \vec{v})$ the $(n_1 + n_2)$-vector $(u_1, \ldots, u_{n_1}, v_1, \ldots, v_{n_2})^T$. An easy consequence of Theorem 1.1 is that if $\vec{x}^{(1)}, \vec{y}^{(1)} \in \mathbf{R}^{n_1}$, and $\vec{x}^{(2)}, \vec{y}^{(2)} \in \mathbf{R}^{n_2}$, and $\vec{x}^{(i)}$ is majorized (submajorized) by $\vec{y}^{(i)}$, $i = 1, 2$, then $(\vec{x}^{(1)}, \vec{x}^{(2)})$ is majorized (submajorized) by $(\vec{y}^{(1)}, \vec{y}^{(2)})$. In particular,

$$\vec{x} \prec \vec{y} \quad \text{if and only if} \quad (\vec{x}, \vec{z}) \prec (\vec{y}, \vec{z}). \tag{1.17}$$

For the statement of a basic theorem on majorization we need the notion of double stochasticity of a matrix. An n-square matrix $A = (a_{ij})$ is said to be *doubly stochastic* if all entries are nonnegative and each column and each row sums to one:

$$a_{ij} \geqslant 0, \qquad i, j = 1, \ldots, n, \tag{1.18}$$

$$\sum_{i=1}^{n} a_{ij} = 1, \qquad j = 1, \ldots, n, \tag{1.19}$$

$$\sum_{j=1}^{n} a_{ij} = 1, \qquad i = 1, \ldots, n. \tag{1.20}$$

The condition (1.18) means

$$(\textit{positivity-preserving:}) \quad A\vec{x} \geqslant 0 \quad \text{whenever} \quad \vec{x} \geqslant 0, \tag{1.18$'$}$$

while (1.19) and (1.20) are equivalent, respectively, to

$$(\textit{trace-preserving:}) \quad \operatorname{tr}(A\vec{x}) = \operatorname{tr}(\vec{x}) \qquad \text{for all } \vec{x}, \tag{1.19$'$}$$

$$(\textit{unital:}) \qquad A\vec{e} = \vec{e}. \tag{1.20$'$}$$

A matrix is trace-preserving if and only if its adjoint is unital. The class of all n-square doubly stochastic matrices is closed under matrix multiplication, adjoint formation, and convex combination; if A, B are doubly stochastic, so

are A^*, AB, and $tA+(1-t)B$ for all $0 \leqslant t \leqslant 1$. Any permutation matrix is doubly stochastic.

THEOREM 1.3. *The following conditions for* $\vec{x}$, $\vec{y} \in \mathbf{R}^n$ *are mutually equivalent*:

(i) $\vec{x} \prec \vec{y}$.

(ii) *There exist a finite number of vectors* $\vec{z}^{(0)}, \dots, \vec{z}^{(N)} \in \mathbf{R}^n$ *such that*

$$\vec{y} = \vec{z}^{(0)} \succ \vec{z}^{(1)} \succ \cdots \succ \vec{z}^{(N)} = \vec{x} \tag{1.21}$$

and such that, for all k, $\vec{z}^{(k)}$ *and* $\vec{z}^{(k+1)}$ *differ in two coordinates only.*

(iii) $\vec{x}$ *is a convex combination of coordinate permutations of* $\vec{y}$.

(iv) $\vec{x} = A\vec{y}$ *for some doubly stochastic matrix* A.

Proof. (i) $\Rightarrow$ (ii) by induction on the dimension n. The case $n=1$ has nothing to prove. Suppose the implication (i) $\Rightarrow$ (ii) is true for all cases of dimension less than n. Let $\vec{x}, \vec{y} \in \mathbf{R}^n$ and $\vec{x} \prec \vec{y}$. Since $\vec{x} \sim \vec{x}^{\downarrow}$ and $\vec{y} \sim \vec{y}^{\downarrow}$, and each coordinate permutation is obtained by successive applications of permutations that exchange two coordinates only, we may assume that $\vec{x} = \vec{x}^{\downarrow}$ and $\vec{y} = \vec{y}^{\downarrow}$. Since $\vec{x} \prec \vec{y}$ implies $y_1 \geqslant x_1 \geqslant y_n$, take k such that $y_{k-1} \geqslant x_1 \geqslant y_k$. Find t such that $0 \leqslant t \leqslant 1$ and $x_1 = ty_1 + (1-t)y_k$, and let

$$\vec{z}^{(1)} = \left(x_1, y_2, \dots, y_{k-1}, (1-t)y_1 + ty_k, y_{k+1}, \dots, y_n\right)^T.$$

Then $\vec{z}^{(1)}$ and $\vec{z}^{(0)} = \vec{y}$ differ only in coordinates 1 and k. Since in $\mathbf{R}^2$

$$\left(ty_1 + (1-t)y_k, (1-t)y_1 + ty_k\right)^T \prec \left(y_1, y_k\right)^T,$$

(1.17) implies $\vec{z}^{(1)} \prec \vec{z}^{(0)}$. Claim: the $(n-1)$-vector $\vec{x}' := (x_2, \dots, x_n)^T$ is majorized by the $(n-1)$-vector.

$$\vec{y}' := \left(y_2, \dots, y_{k-1}, (1-t)y_1 + ty_k, y_{k+1}, \dots, y_n\right)^T.$$

For $2 \leqslant l \leqslant k-1$, $y_1 \geqslant \cdots \geqslant y_{k-1} \geqslant x_1 \geqslant x_2 \geqslant \cdots$ implies

$$\sum_{j=2}^{l} x_j \leqslant \sum_{j=2}^{l} y_j,$$

while for $k \leqslant l \leqslant n$

$$\sum_{j=2}^{k-1} y_j + (1-t)y_1 + ty_k + \sum_{j=k+1}^{l} y_j = \sum_{j=1}^{l} y_j - x_1$$

$$\geqslant \sum_{j=1}^{l} x_j - x_1 = \sum_{j=2}^{l} x_j.$$

The last inequality becomes equality when $l = n$. This establishes the claim. By the induction assumption there exist a finite number of vectors $\vec{w}^{(0)}, \dots, \vec{w}^{(N-1)} \in \mathbf{R}^{n-1}$ such that

$$\vec{y}' = \vec{w}^{(0)} \succ \vec{w}^{(1)} \succ \cdots \succ \vec{w}^{(N-1)} = \vec{x}'$$

and such that, for all k, $\vec{w}^{(k-1)}$ and $\vec{w}^{(k)}$ differ in two coordinates only. Now $\vec{z}^{(k)} := (x_1, \vec{w}^{(k-1)})$, $k = 2, \dots, N$, satisfy (1.21).

(ii) $\Rightarrow$ (iii): At each k, $\vec{z}^{(k)} = [tI + (1-t)\Pi]\vec{z}^{(k-1)}$ for some $0 \leqslant t \leqslant 1$ and a permutation matrix Π that interchanges two indices only. Now (iii) follows from the fact that any product of permutation matrices is again a permutation matrix.

(iii) $\Rightarrow$ (iv): This results from the fact that any convex combination of permutation matrices is doubly stochastic.

(iv) $\Rightarrow$ (i): Suppose that $\vec{x} = A\vec{y}$ for a doubly stochastic matrix A. Since A is unital,

$$\vec{x} - t\vec{e} = A(\vec{y} - t\vec{e}) \qquad \text{for all} \quad t \in \mathbf{R}.$$

Then the positivity-preserving property of A implies

$$|\vec{x} - t\vec{e}| \leqslant A|\vec{y} - t\vec{e}| \qquad \text{for all} \quad t \in \mathbf{R}.$$

Finally since A is trace-preserving,

$$\operatorname{tr}|\vec{x} - t\vec{e}| \leqslant \operatorname{tr} A|\vec{y} - t\vec{e}| = \operatorname{tr}|\vec{y} - t\vec{e}| \qquad \text{for all} \quad t \in \mathbf{R}.$$

Now (i) results from Corollary 1.2. ∎

The above proof also shows that the doubly stochastic matrix in Theorem 1.3(iv) can be required to be the product of at most $n-1$ matrices of the form $tI + (1-t)\Pi$, where $0 \leqslant t \leqslant 1$ and Π is a permutation matrix that just interchanges two coordinates.

Another special class of doubly stochastic matrices is related to unitary matrices: if $U=(u_{ij})$ is unitary, the matrix $(|u_{ij}|^2)$ is doubly stochastic. In particular, any real unitary matrix produces a doubly stochastic matrix in this way.

THEOREM 1.4. *The doubly stochastic matrix A in Theorem 1.3(iv) can be required to have the form*

$$A=\left(|u_{ij}|^2\right) \qquad \text{for a real unitary matrix } U=\left(u_{ij}\right). \tag{1.22}$$

Proof by induction on the dimension n. The case $n=1$ is trivial. Suppose that the assertion is true for all cases of dimension less than n. Remark that $\vec{x}=A\vec{y}$ with A satisfying (1.22) means that $U\operatorname{diag}(\vec{y})U^*$ has $\vec{x}$ on its diagonal, where $\operatorname{diag}(\vec{y})$ is the diagonal matrix with diagonal vector $\vec{y}$. In the proof of the implication (i) $\Rightarrow$ (ii) of Theorem 1.3, define a real unitary matrix V by

$$v_{11}=v_{kk}=\sqrt{t}\,, \qquad v_{1k}=\sqrt{1-t}\,, \qquad v_{k1}=-\sqrt{1-t}\,,$$

$$v_{ij}=\delta_{ij} \qquad \text{for other } i,j.$$

Then $V\operatorname{diag}(\vec{y})V^*$ has $\vec{z}^{(1)}$ on its diagonal, and the other nonzero entries appear at $(1,k)$ and $(k,1)$ positions. Since the $(n-1)$-vector $\vec{y}'=(y_2,\dots,y_{k-1},(1-t)y_1+ty_k,y_{k+1},\dots,y_n)^T$ is proved to majorize $\vec{x}'=(x_2,\dots,x_n)$ according to the induction assumption, there exists an $(n-1)$-square real unitary matrix W such that $W\operatorname{diag}(\vec{y}')W^*$ has $\vec{x}'$ on its diagonal. The n-square real unitary matrix

$$U:=\begin{pmatrix}1 & 0\\ 0 & W\end{pmatrix}\cdot V$$

meets the requirement. ∎

Let us add one more interesting example of natural occurrence of majorization.

EXAMPLE 4. Let A be a Hermitian matrix. Denote by $\vec{\lambda}(A)$ the n-vector of its eigenvalues, arranged in any order, and by $\vec{\delta}(A)$ the diagonal vector. According to the spectral theory there exists a unitary matrix $V=(v_{ij})$ such that

$$A=V\operatorname{diag}\left(\vec{\lambda}(A)\right)V^*. \tag{1.23}$$

With $P = (|v_{ij}|^2)$ it follows from (1.23)

$$\vec{\delta}(A) = P\vec{\lambda}(A).$$

Since P is doubly stochastic, $\vec{\delta}(A) \prec \vec{\lambda}(A)$.

Combination of Corollary 1.2 and Theorem 1.3 yields that a linear map T defined on $\mathrm{span}(\vec{y}, \vec{e})$, the linear span of $\vec{y}$ and $\vec{e}$, by

$$T(s\vec{y} + t\vec{e}) = s\vec{x} + t\vec{e}, \qquad s, t \in \mathbf{R},$$

is extended to a *stochastic* linear map, that is, a positivity-preserving and trace-preserving linear map, if and only if (1.15) holds. The following theorem gives a general treatment of this type.

THEOREM 1.5. *Let $\mathcal{M}$ be a (real linear) subspace of $\mathbf{R}^n$, and T a (real) linear map from $\mathcal{M}$ to $\mathbf{R}^m$. Then T admits a stochastic linear extension, that is, there exists a stochastic linear map A from $\mathbf{R}^n$ to $\mathbf{R}^m$ such that $A\vec{x} = T\vec{x}$ for all $\vec{x} \in \mathcal{M}$ if and only if*

$$\mathrm{tr}\left(\bigvee_{j=1}^{N} T\vec{x}^{(j)}\right) \leqslant \mathrm{tr}\left(\bigvee_{j=1}^{N} \vec{x}^{(j)}\right) \qquad \textit{for all} \quad \vec{x}^{(j)} \in \mathcal{M}. \tag{1.24}$$

Here $\vee_{j=1}^{N} \vec{x}^{(j)}$, for instance, is the coordinatewise maximum of $\vec{x}^{(1)}, \ldots, \vec{x}^{(N)}$.

Proof. Suppose first that T admits a stochastic linear extension A. Then

$$\begin{aligned}
\mathrm{tr}\left(\bigvee_{j=1}^{N} T\vec{x}^{(j)}\right) &= \mathrm{tr}\left(\bigvee_{j=1}^{N} A\vec{x}^{(j)}\right) \\
&\leqslant \mathrm{tr}\left(A\left(\bigvee_{j=1}^{N} \vec{x}^{(j)}\right)\right) \qquad \text{(positivity-preserving)} \\
&= \mathrm{tr}\left(\bigvee_{j=1}^{N} \vec{x}^{(j)}\right) \qquad \text{(trace-preserving)}.
\end{aligned}$$

Suppose conversely that T satisfies (1.24). Identifying the space $\mathbf{R}^{nm}$ with the direct sum $\overbrace{\mathbf{R}^n \oplus \cdots \oplus \mathbf{R}^n}^{m}$, let us denote a vector in $\mathbf{R}^{nm}$ by $\vec{\vec{x}} = (\vec{x}^{(1)}, \ldots, \vec{x}^{(m)})$

with $\vec{x}^{(j)} \in \mathbf{R}^n$, $j = 1, \ldots, m$. Define a real-valued function(al) $p(\vec{x})$ on $\mathbf{R}^{nm}$ by

$$p(\vec{x}) = \operatorname{tr}\left(\bigvee_{j=1}^{m} \vec{x}^{(j)} \right). \tag{1.25}$$

p is *subadditive* and *positively homogeneous*, that is,

$$p(\vec{x}+\vec{y}) \leqslant p(\vec{x}) + p(\vec{y}) \quad \text{and} \quad p(t\vec{x}) = tp(\vec{x}) \quad \text{for } t \geqslant 0.$$

Let $\hat{\mathscr{M}}$ be the subspace of all $\vec{x} = (\vec{x}^{(1)}, \ldots, \vec{x}^{(m)})$ with $\vec{x}^{(j)} \in \mathscr{M}$, $j = 1, \ldots, m$, and define a linear function(al) ϕ on $\hat{\mathscr{M}}$ by

$$\phi(\vec{x}) = \sum_{j=1}^{m} \langle T\vec{x}^{(j)}, \vec{e}_{(j)} \rangle,$$

where

$$\vec{e}_{(j)} = (0, \ldots, 0, \overset{(j)}{1}, 0, \ldots, 0)^T \text{ of } \mathbf{R}^m, \; j = 1, \ldots, m.$$

Then the condition (1.24) implies

$$\begin{aligned} \phi(\vec{x}) &\leqslant \left\langle \bigvee_{j=1}^{m} T\vec{x}^{(j)}, \sum_{j=1}^{m} \vec{e}_{(j)} \right\rangle = \operatorname{tr}\left(\bigvee_{j=1}^{m} T\vec{x}^{(j)} \right) \\ &\leqslant \operatorname{tr}\left(\bigvee_{j=1}^{m} \vec{x}^{(j)} \right) = p(\vec{x}). \end{aligned}$$

According to a theorem of the Hahn-Banach type, ϕ can be extended to a linear function(al) $\hat{\phi}$ on the whole space $\mathbf{R}^{nm}$ such that $\hat{\phi}$ coincides with ϕ on $\hat{\mathscr{M}}$, and

$$\hat{\phi}(\vec{x}) \leqslant p(\vec{x}) = \operatorname{tr}\left(\bigvee_{j=1}^{m} \vec{x}^{(j)} \right) \qquad \text{for all} \quad \vec{x} \in \mathbf{R}^{nm}. \tag{1.26}$$

There exist uniquely $\vec{a}^{(j)} \in \mathbf{R}^n$, $j = 1, \ldots, m$, such that

$$\hat{\phi}(\vec{x}) = \sum_{j=1}^{m} \langle \vec{x}^{(j)}, \vec{a}^{(j)} \rangle \qquad \text{for all} \quad \vec{x} \in \mathbf{R}^{nm}. \tag{1.27}$$

Define a linear map A from $\mathbf{R}^n$ to $\mathbf{R}^m$ by

$$A\vec{x} = \left(\langle \vec{x}, \vec{a}^{(1)} \rangle, \dots, \langle \vec{x}, \vec{a}^{(m)} \rangle\right)^T. \tag{1.28}$$

Given $\vec{x} \in \mathscr{M}$ and $1 \leqslant k \leqslant m$, consider the vector

$$\vec{\vec{x}}_{(k)} = (0, \dots, 0, \overset{(k)}{\vec{x}}, 0, \dots, 0).$$

Then it follows from the definitions (1.27) and (1.28) that

$$\begin{aligned}\langle T\vec{x}, \vec{e}_{(k)} \rangle &= \phi\left(\vec{\vec{x}}_{(k)}\right) = \hat{\phi}\left(\vec{\vec{x}}_{(k)}\right) \\ &= \langle \vec{x}, \vec{a}^{(k)} \rangle = \langle A\vec{x}, \vec{e}_{(k)} \rangle.\end{aligned}$$

Since $\vec{x} \in \mathscr{M}$ and k are arbitrary, this shows that A coincides with T on $\mathscr{M}$. A is positivity-preserving. In fact, for $0 \leqslant \vec{x} \in \mathbf{R}^n$, consider $\vec{\vec{x}}_{(k)}$ as above; then (1.26) implies

$$\begin{aligned}\langle -\vec{x}, \vec{a}^{(k)} \rangle = \hat{\phi}\left(-\vec{\vec{x}}_{(k)}\right) &\leqslant p\left(-\vec{\vec{x}}_{(k)}\right) \\ &= \operatorname{tr}(\vec{x}^{\,-}) = 0,\end{aligned}$$

which shows $\vec{a}^{(k)} \geqslant 0$. Finally A is trace-preserving. For $\vec{x} \in \mathbf{R}^n$, consider $\vec{\vec{x}} = (\vec{x}, \dots, \vec{x})$. Then (1.26) implies

$$\operatorname{tr}(A\vec{x}) = \hat{\phi}(\vec{\vec{x}}) \leqslant p(\vec{\vec{x}}) = \operatorname{tr}(\vec{x}),$$

and similarly

$$\operatorname{tr}\left(A(-\vec{x})\right) \leqslant -\operatorname{tr}(\vec{x}).$$

■

The case of $\dim(\mathscr{M}) = 2$ gives a direct generalization of Corollary 1.2.

COROLLARY 1.6. *Let $0 \leqslant \vec{x}^{(i)} \in \mathbf{R}^m$ and $0 \leqslant \vec{y}^{(i)} \in \mathbf{R}^n$, $i = 1, 2$. There exists a stochastic linear map A such that $A\vec{y}^{(i)} = \vec{x}^{(i)}$, $i = 1, 2$, if and only if $\operatorname{tr}(\vec{y}^{(i)}) = \operatorname{tr}(\vec{x}^{(i)})$, $i = 1, 2$, and*

$$\operatorname{tr}|\vec{x}^{(1)} + t\vec{x}^{(2)}| \leqslant \operatorname{tr}|\vec{y}^{(1)} + t\vec{y}^{(2)}| \qquad \text{for all} \quad t \in \mathbf{R}. \tag{1.29}$$

Proof. "Only if" is proved just as in Theorem 1.5. Suppose that $\mathrm{tr}(\vec{y}^{(i)}) = \mathrm{tr}(\vec{x}^{(i)})$, $i = 1,2$, and (1.29) is fulfilled. An immediate consequence is that

$$T\big(\alpha\vec{y}^{(1)} + \beta\vec{y}^{(2)}\big) = \alpha\vec{x}^{(1)} + \beta\vec{x}^{(2)} \qquad \text{for all} \quad \alpha, \beta \in \mathbf{R}$$

well defines a linear map from $\mathrm{span}(\vec{y}^{(1)}, \vec{y}^{(2)})$ to $\mathbf{R}^m$. To appeal to Theorem 1.5, it is enough to show that for any $\alpha_j, \beta_j \in \mathbf{R}$, $j = 1, \dots, N$,

$$\mathrm{tr}\left[\bigvee_{j=1}^{N}\big(\alpha_j\vec{x}^{(1)} + \beta_j\vec{x}^{(2)}\big)\right] \leqslant \mathrm{tr}\left[\bigvee_{j=1}^{N}\big(\alpha_j\vec{y}^{(1)} + \beta_j\vec{y}^{(2)}\big)\right]. \tag{1.30}$$

The function

$$f(s) = \bigvee_{j=1}^{N}(\alpha_j s + \beta_j)$$

is continuous and convex, and there exist $s_1 < \cdots < s_M$ such that $f(s)$ is linear on $(-\infty, s_1]$, $[s_M, \infty)$, and each $[s_k, s_{k+1}]$, $k = 1, \dots, M-1$. Let $s_0 = s_1 - 1$ and $s_\infty = s_M + 1$. The convex function $f(s)$ has right derivative $f'(s)$ which is a nondecreasing function. Simple computation will show the following representation:

$$f(s) = f(s_0) + (s - s_0)f'(s_0) + \int_{s_0}^{s_\infty}(s-t)^+\, df'(t);$$

then for any $u,\ v \geqslant 0$

$$\bigvee_{j=1}^{N}(\alpha_j u + \beta_j v) = f(s_0)v + (u - s_0 v)f'(s_0) + \int_{s_0}^{s_\infty}(u - tv)^+\, df'(t). \tag{1.31}$$

Since $\vec{x}^{(i)} \geqslant 0$, $i = 1,2$, in the sense of a vector-valued integral, (1.31) implies

$$\bigvee_{j=1}^{N}\big(\alpha_j\vec{x}^{(1)} + \beta_j\vec{x}^{(2)}\big)$$

$$= f(s_0)\vec{x}^{(2)} + f'(s_0)\big(\vec{x}^{(1)} - s_0\vec{x}^{(2)}\big) + \int_{s_0}^{s_\infty}\big(\vec{x}^{(1)} - t\vec{x}^{(2)}\big)^+\, df'(t),$$

and consequently

$$\operatorname{tr}\left[\bigvee_{j=1}^{N}\left(\alpha_j\vec{x}^{(1)}+\beta_j\vec{x}^{(2)}\right)\right]=f(s_0)\operatorname{tr}(\vec{x}^{(2)})+f'(s_0)\left\{\operatorname{tr}(\vec{x}^{(1)})-s_0\operatorname{tr}(\vec{x}^{(2)})\right\}$$

$$+\int_{s_0}^{s_\infty}\operatorname{tr}\left(\vec{x}^{(1)}-t\vec{x}^{(2)}\right)^{+}df'(t). \tag{1.32}$$

The corresponding identity holds with $\vec{y}^{(i)}$ instead of $\vec{x}^{(i)}$, $i=1,2$. Now since (1.29) together with $\operatorname{tr}(\vec{x}^{(i)})=\operatorname{tr}(\vec{y}^{(i)})$, $i=1,2$, implies, as in the proof of Corollary 1.2,

$$\operatorname{tr}\left(\vec{x}^{(1)}-t\vec{x}^{(2)}\right)^{+}\leqslant\operatorname{tr}\left(\vec{y}^{(1)}-t\vec{y}^{(2)}\right)^{+},$$

the inequality (1.32) and the corresponding one with $\vec{y}^{(i)}$, $i=1,2$, yield (1.30), because $f'(t)$ is increasing. ■

Now let us turn to a study of sub- or supermajorization. Remark first of all that

$$\vec{x}\underset{\cdot}{\prec}\vec{y}\qquad\text{if}\quad\vec{x}\leqslant\vec{y}. \tag{1.33}$$

THEOREM 1.7. *Suppose that* $\vec{u}\underset{\cdot}{\prec}\vec{y}$ *and* $\vec{v}\overset{\cdot}{\prec}\vec{y}$. *If* $\vec{u}\leqslant\vec{v}$, *then there exists* $\vec{x}$ *such that* $\vec{x}\prec\vec{y}$ *and* $\vec{u}\leqslant\vec{x}\leqslant\vec{v}$.

Proof by induction on the dimension n. The case $n=1$ is trivially true. Suppose that the assertion is true for all the cases of dimension less than n. Let

$$\vec{x}(t):=(1-t)\vec{u}+t\vec{v}\qquad\text{for}\quad 0\leqslant t\leqslant 1.$$

Then $\vec{u}\leqslant\vec{x}(t)\leqslant\vec{v}$, and $\vec{x}(0)=\vec{u}\underset{\cdot}{\prec}\vec{y}$. Let t_0 be the maximum of all t such that $\vec{x}(t)\underset{\cdot}{\prec}\vec{y}$. If $t_0=1$, then $\vec{x}(1)=\vec{v}$ is submajorized and supermajorized by $\vec{y}$; hence $\vec{x}:=\vec{v}$ meets the requirement. If $t_0<1$, in view of the maximum property of t_0, there exists $1\leqslant k\leqslant n$ such that

$$\sum_{j=1}^{k}x(t_0)_j^{\cdot}=\sum_{j=1}^{k}y_j^{\cdot}. \tag{1.34}$$

We may assume here $\vec{x}(t_0)=\vec{x}(t_0)^{\cdot}$. Then the k-vector $(x(t_0)_1,\ldots,x(t_0)_k)^T$ is

majorized by the k-vector $(y_1^{\bullet},\ldots,y_k^{\bullet})^T$. Consider the $(n-k)$-vectors

$$\vec{y}' := (y_{k+1}^{\bullet},\ldots,y_n^{\bullet})^T, \qquad \vec{u}' := (u_{k+1},\ldots,u_n)^T,$$

$$\vec{v}' := (v_{k+1},\ldots,v_n)^T \quad \text{and} \quad \vec{w}' := (x(t_0)_{k+1},\ldots,x(t_0)_n)^T.$$

Since (1.34) implies

$$\vec{w}' \underset{\bullet}{\prec} \vec{y}', \vec{v}' \overset{\bullet}{\prec} \vec{y}', \quad \text{and} \quad \vec{u}' \leqslant \vec{w}' \leqslant \vec{v}',$$

according to the induction assumption applied to the triple $\vec{y}'$, $\vec{w}'$, and $\vec{v}'$, there exists an $(n-k)$-vector $\vec{x}'$ such that $\vec{x}' \prec \vec{y}'$ and $\vec{w}' \leqslant \vec{x}' \leqslant \vec{v}'$. Then the n-vector $\vec{x} := (x(t_0)_1,\ldots,x(t_0)_k,\vec{x}')$ meets the requirement. ∎

Before stating a corollary, let us introduce some notions used subsequently. A square matrix $B=(b_{ij})$ is said to be *doubly substochastic* if all entries are nonnegative and there exists a doubly stochastic matrix $A=(a_{ij})$ such that

$$0 \leqslant b_{ij} \leqslant a_{ij} \qquad \text{for all } i \text{ and } j. \tag{1.35}$$

Correspondingly a square matrix $C=(c_{ij})$ is said to be *doubly superstochastic* if there exists a doubly stochastic matrix $A=(a_{ij})$ such that

$$c_{ij} \geqslant a_{ij} \qquad \text{for all } i \text{ and } j. \tag{1.36}$$

COROLLARY 1.8. *The following conditions for $\vec{u}$, $\vec{y} \in \mathbf{R}^n$ are mutually equivalent*:

(i) $\vec{u} \underset{\bullet}{\prec} \vec{y}$.

(ii) *There exists $\vec{x}$ such that $\vec{x} \prec \vec{y}$ and $\vec{u} \leqslant \vec{x}$.*

(iii) *(Under the assumption $\vec{y} \geqslant 0$, $\vec{u} \geqslant 0$) there exists a doubly substochastic matrix B such that $\vec{u} = B\vec{y}$.*

Proof. (i) $\Rightarrow$ (ii): For sufficiently large t the vector $t\vec{e}$ is supermajorized by $\vec{y}$ and $\vec{u} \leqslant t\vec{e}$, and Theorem 1.7 can be applied.

(ii) $\Rightarrow$ (i) follows from (1.33).

(ii) $\Rightarrow$ (iii): According to Theorem 1.3, $\vec{x} = A\vec{y}$ for a doubly stochastic matrix A. Then the matrix $B := \operatorname{diag}(\vec{u}/\vec{x}) \cdot A$ meets the requirement, where $\vec{u}/\vec{x}$ is defined by coordinatewise division with the convention $0/0 = 0$.

(iii) $\Rightarrow$ (ii) follows from the definition of substochasticity via Theorem 1.3. ∎

A corresponding assertion holds with supermajorization and double superstochasticity.

COROLLARY 1.9. *Suppose that* $\vec{x}$, $\vec{y} \geqslant 0$ *and* $\vec{y} = \sum_{j=1}^{k} \vec{y}^{(j)}$ *with* $\vec{y}^{(j)} \geqslant 0$, $j = 1, \dots, k$. *If* $\vec{x}$ *is majorized (sub- or supermajorized) by* $\vec{y}$, *then there exists a decomposition* $\vec{x} = \sum_{j=1}^{k} \vec{x}^{(j)}$ *such that* $\vec{x}^{(j)} \geqslant 0$ *and* $\vec{x}^{(j)}$ *is majorized (sub- or supermajorized) by* $\vec{y}^{(j)}$, $j = 1, \dots, k$.

Proof. According to Theorem 1.3 and Corollary 1.9, $\vec{x} = A\vec{y}$ with doubly stochastic (sub- or superstochastic) A. Let $\vec{x}^{(j)} = A\vec{y}^{(j)}$, $j = 1, \dots, k$. ∎

NOTE. A history of development of the notion of majorization is described in the monograph [51]. Physicists use the reversed notation $\vec{x} \succ \vec{y}$ to denote that $\vec{x}$ is majorized by $\vec{y}$ in our sense; they say that $\vec{x}$ is more chaotic than $\vec{y}$ (see [2]). The basic result (Theorem 1.3) is due to Hardy, Littlewood, and Pólya [35] and also to Rado [64]. Theorem 1.4 is in Horn [36]. Theorem 1.5 is found in Alberti and Uhlmann [1, 2]; F. Niiro has communicated an equivalent form. For related topics, see [8], [68], [25, 26], and [56]. Corollary 1.6 is proved by Ruch, Schranner and Seligman [66] by a different method. For Theorem 1.7 and its relatives, see Fan [23] and Chong [15, 16].

2. *Isotone Maps and Algebraic Operations*

$\mathbf{R}^n$ (and $\mathbf{R}^m$) is provided with the usual order structure $\leqslant$. A map Φ from (a subset of) $\mathbf{R}^n$ to $\mathbf{R}^m$ is said to be *monotone increasing* if it is order-preserving, that is, if

$$\Phi(\vec{x}) \leqslant \Phi(\vec{y}) \quad \text{whenever} \quad \vec{x} \leqslant \vec{y}. \tag{2.1}$$

Φ is said to be *monotone decreasing* if $-\Phi$ is monotone increasing. When Φ is linear, it is monotone increasing if and only if it is positivity-preserving. Another notion of importance is convexity. Φ is said to be *convex* if

$$\Phi(t\vec{x} + (1-t)\vec{y}) \leqslant t\Phi(\vec{x}) + (1-t)\Phi(\vec{y}) \qquad \text{for} \quad 0 \leqslant t \leqslant 1. \tag{2.2}$$

Φ is said to be *concave* if $-\Phi$ is convex.

$\mathbf{R}^n$ (and $\mathbf{R}^m$) is also provided with the preorder structure of majorization. A map Φ from (a subset of) $\mathbf{R}^n$ to $\mathbf{R}^m$ is said to be *isotone* if

$$\Phi(\vec{x}) \prec \Phi(\vec{y}) \quad \text{whenever} \quad \vec{x} \prec \vec{y}. \tag{2.3}$$

Very often the word "*Schur-convex*" is used instead of "isotone." This notion admits two natural specializations: Φ is said to be *strongly isotone* if

$$\Phi(\vec{x}) \prec_{w} \Phi(\vec{y}) \quad \text{whenever} \quad \vec{x} \prec_{w} \vec{y}, \tag{2.4}$$

while it is *strictly isotone* if

$$\Phi(\vec{x}) \prec \Phi(\vec{y}) \quad \text{whenever} \quad \vec{x} \prec \vec{y}. \tag{2.5}$$

In this section the domain of definition for a map is always assumed to be a convex set that is invariant under all coordinate permutations.

THEOREM 2.1. *If a map Φ from $\mathbf{R}^n$ to $\mathbf{R}^m$ is convex and if for any permutation matrix Π (of order n) there exists a permutation matrix $\hat{\Pi}$ (of order m) such that*

$$\hat{\Pi}\Phi(\vec{x}) = \Phi(\Pi\vec{x}) \qquad \textit{for all } \vec{x}, \tag{2.6}$$

then Φ is isotone. It becomes strongly isotone if, in addition, it is monotone increasing.

Proof. Let $\vec{x} \prec \vec{y}$ in $\mathbf{R}^n$. According to Theorem 1.3 there exist $t_j > 0$ and n-square permutation matrices $\Pi^{(j)}$, $j = 1, \dots, N$, such that

$$\sum_{j=1}^{N} t_j = 1 \quad \text{and} \quad \vec{x} = \sum_{j=1}^{N} t_j \Pi^{(j)} \vec{y}.$$

Then

$$\begin{aligned} \Phi(\vec{x}) &= \Phi\left(\sum_{j=1}^{N} t_j \Pi^{(j)} \vec{y} \right) \\ &\leqslant \sum_{j=1}^{N} t_j \Phi(\Pi^{(j)} \vec{y}) \qquad \text{(convexity)} \\ &= \sum_{j=1}^{N} t_j \hat{\Pi}^{(j)} \Phi(\vec{y}) \qquad \text{by (2.6)}, \end{aligned}$$

which yields, again by Theorem 1.3, $\Phi(\vec{x}) \prec_{w} \Phi(\vec{y})$.

Suppose now that, in addition, Φ is monotone increasing. If $\vec{u} \prec_w \vec{y}$ in $\mathbf{R}^n$, by Corollary 1.8 there exists $\vec{x}$ such that $\vec{u} \leqslant \vec{x} \prec \vec{y}$. Then $\Phi(\vec{x}) \prec_w \Phi(\vec{y})$ and $\Phi(\vec{u}) \leqslant \Phi(\vec{x})$ by monotony, hence $\Phi(\vec{u}) \prec_w \Phi(\vec{y})$. ■

Any real-valued function $f(t)$ defined on a (finite or infinite) interval Ω of the real line can induce a map, denoted by the same letter, from $\overbrace{\Omega \times \cdots \times \Omega}^{n}$ to $\mathbf{R}^n$ by

$$f(\vec{x}) := \big(f(x_1), \dots, f(x_n)\big). \tag{2.7}$$

Corollary 2.2. *If $f(t)$ is convex, the map $f(\vec{x})$ is isotone. If, in addition, $f(t)$ is monotone increasing, $f(\vec{x})$ is strongly isotone.*

Proof. The convexity of $f(t)$ implies that of the map $f(\vec{x})$, and (2.6) is satisfied with $\hat{\Pi} = \Pi$. ■

Take $f(t) = |t|$, t^2, or t^+ to obtain the following:

$$|\vec{x}| \prec_w |\vec{y}| \quad \text{whenever} \quad \vec{x} \prec \vec{y}; \tag{2.8}$$

$$\vec{x}^2 \prec_w \vec{y}^2 \quad \text{whenever} \quad \vec{x} \prec \vec{y}; \tag{2.9}$$

$$\vec{x}^+ \prec_w \vec{y}^+ \quad \text{whenever} \quad \vec{x} \prec_w \vec{y}. \tag{2.10}$$

When $m = 1$, $\Phi(\vec{x})$ is a real-valued function, and the condition (2.6) implies its *permutation-invariance*:

$$\Phi(\Pi\vec{x}) = \Phi(\vec{x}) \qquad \text{for all permutation matrices } \Pi.$$

This condition is also necessary for the isotony of the function Φ, because $\vec{x} \sim \vec{y}$ implies $\Phi(\vec{x}) = \Phi(\vec{y})$.

Corollary 2.3. *If $\Psi(\vec{x})$ is a convex function on $\mathbf{R}^n$, then the function defined by*

$$\Phi(\vec{x}) := \max_{\Pi} \Psi(\Pi\vec{x}),$$

Π running over all permutation matrices, is isotone. If, in addition, Ψ is monotone increasing, Φ is strongly isotone.

In fact, Φ is convex and permutation-invariant.

Corollary 2.4. *If $f(t)$ is a convex function on* $\mathbf{R}$, *for any* $1 \leqslant k \leqslant n$ *the function on* $\mathbf{R}^n$ *defined by*

$$\Phi^{(k)}(\vec{x}) = \max_{\pi \in \mathscr{S}_n} \sum_{j=1}^{k} f(x_{\pi_j}),$$

where $\mathscr{S}_n$ *is the set of all permutation of order* n, *is isotone. In particular*

$$\Phi^{(n)}(\vec{x}) := \sum_{j=1}^{n} f(x_j) = \mathrm{tr}[f(\vec{x})]$$

is isotone. If, in addition, $f(t)$ *is monotone increasing, these functions* $\Phi^{(k)}$ *are strongly isotone.*

Remark that when $f(t) = t$,

$$\Phi^{(k)}(\vec{x}) = \sum_{j=1}^{k} x_j^{\cdot}, \qquad k = 1, \dots, n-1,$$

$$\Phi^{(n)}(\vec{x}) = \mathrm{tr}(\vec{x}).$$

Therefore the submajorization $\vec{x} \prec_{\cdot} \vec{y}$ is recovered by using n strongly isotone function $\Phi^{(k)}$, $k = 1, \dots, n$.

Let us present some simple examples of isotone functions along the line of Corollary 2.4:

Example 1. The *deviation function* is isotone:

$$V(\vec{x}) := \sum_{j=1}^{n} \left(x_j - \frac{\mathrm{tr}(\vec{x})}{n} \right)^2.$$

Example 2. The *entropy function* is isotone on the set of positive vectors:

$$H(\vec{x}) := \sum_{j=1}^{n} x_j \log x_j \qquad (\text{with} \quad 0.\infty = 0).$$

THEOREM 2.5. *Suppose that a real-valued function* $\Phi(\vec{x}) = \Phi(x_1,\ldots,x_n)$ *is differentiable in any argument* x_j, *and write* $\Phi_{(j)} = (\partial/\partial x_j)\Phi$, $j = 1,\ldots,n$. *Then* Φ *is isotone if and only if the following two conditions are satisfied*:

(a) Φ *is permutation-invariant.*
(b) $(x_i - x_j)\{\Phi_{(i)}(\vec{x}) - \Phi_{(j)}(\vec{x})\} \geqslant 0$ *for all* $\vec{x}$ *and* i, j.

Proof. Suppose first that Φ is isotone. Then (a) is obvious, and it suffices to prove (b) with $i = 1$ and $j = 2$. Given $\vec{x} = (x_1,\ldots,x_n)$, define, for $0 \leqslant t \leqslant 1$,

$$\vec{x}(t) = ((1-t)x_1 + tx_2,\ tx_1 + (1-t)x_2,\ x_3,\ldots,x_n)^T. \tag{2.11}$$

Since $\vec{x}(t) \prec \vec{x} = \vec{x}(0)$, the isotony implies

$$0 \geqslant \frac{d}{dt}\Phi(\vec{x}(t))\bigg|_{t=0} = -(x_1 - x_2)\{\Phi_{(1)}(\vec{x}) - \Phi_{(2)}(\vec{x})\},$$

which proves (b) for $i = 1$ and $j = 2$.

Suppose conversely that (a) and (b) are satisfied. In view of Theorem 1.3 and the permutation invariance (a), Φ will be isotone if $\Phi(\vec{u}) \leqslant \Phi(\vec{x})$ whenever

$$\vec{x} = (x_1,\ldots,x_n)^T \quad \text{and} \quad \vec{u} = ((1-s)x_1 + sx_2,\ sx_1 + (1-s)x_2,\ x_3,\ldots,x_n)^T$$

for some $0 \leqslant s \leqslant \frac{1}{2}$. Define $\vec{x}(t)$ by (2.11). Then

$$\begin{aligned}
\Phi(\vec{u}) - \Phi(\vec{x}) &= \int_0^s \frac{d}{dt}\Phi(\vec{x}(t))\,dt \\
&= -\int_0^s (x_1 - x_2)\{\Phi_{(1)}(\vec{x}(t)) - \Phi_{(2)}(\vec{x}(t))\}\,dt \\
&= -\int_0^s \frac{x(t)_1 - x(t)_2}{1-2t}\{\Phi_{(1)}(\vec{x}(t)) - \Phi_{(2)}(\vec{x}(t))\}\,dt \\
&\leqslant 0 \qquad \left(\text{by (b) and } 0 \leqslant s \leqslant \tfrac{1}{2}\right).
\end{aligned}$$

This completes the proof. ∎

This theorem has a lot of applications. Let us cite only one. The kth *elementary symmetric function* $S_k(\vec{x})$ is defined by

$$S_k(\vec{x}) \equiv S_k(x_1,\dots,x_n) := \sum x_{i_1} \cdots x_{i_k}, \tag{2.12}$$

where the summation is extended over all choices $i_1 < i_2 < \dots < i_k$.

EXAMPLE 3. For any $1 \leqslant k \leqslant n$ the function $S_k(\vec{x})$ is antiisotone, that is, $-S_k(\vec{x})$ is isotone, on the set of positive vectors.

In fact, $\Phi(\vec{x}) := -S_k(\vec{x})$ is permutation invariant, and

$$\begin{aligned}
&(x_1 - x_2)\{\Phi_{(1)}(\vec{x}) - \Phi_{(2)}(\vec{x})\}\\
&\quad = -(x_1 - x_2)\{S_{k-1}(x_2, x_3,\dots,x_n) - S_{k-1}(x_1, x_3,\dots,x_n)\}\\
&\quad = (x_1 - x_2)^2 S_{k-2}(x_3,\dots,x_n) \geqslant 0.
\end{aligned}$$

The particular case $k = n$ yields

$$\prod_{j=1}^{n} x_j \geqslant \prod_{j=1}^{n} y_j \quad \text{whenever} \quad \vec{x}, \vec{y} \geqslant 0 \text{ and } \vec{x} \prec \vec{y}. \tag{2.13}$$

The inequality (2.13) produces the Hadamard determinant theorem: if $A = (a_{ij})$ is a positive (definite) matrix,

$$\det(A) \leqslant \prod_{j=1}^{n} a_{jj}. \tag{2.14}$$

In fact, it is pointed out in Section 1 that the diagonal vector $\vec{\delta}(A)$ is majorized by the eigenvalue vector $\vec{\lambda}(A)$. Since both $\vec{\delta}(A)$ and $\vec{\lambda}(A)$ are positive, by (2.13)

$$\det(A) = \prod_{j=1}^{n} \lambda_j(A) \leqslant \prod_{j=1}^{n} a_{jj}.$$

Isotony turns out to be a too restrictive condition for a linear map.

THEOREM 2.6. *Let A be a linear map from* $\mathbf{R}^n$ *to* $\mathbf{R}^m$. *Then the following conditions are mutually equivalent*:

(i) *A is strictly isotone.*

(ii) *A is isotone.*

(iii) *$A\vec{x} \sim A\vec{y}$ whenever $\vec{x} \sim \vec{y}$.*

(iv) *For any permutation matrix Π of order n there exists a permutation matrix $\hat{\Pi}$ of order m such that*

$$A\Pi = \hat{\Pi}A. \tag{2.15}$$

Proof. The implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial. Since A is convex, (iv) $\Rightarrow$ (i) follows Theorem 2.1.

(iii) $\Rightarrow$ (iv): Let the linear map A be represented in the form

$$A\vec{x} = \left(\langle \vec{x}, \vec{a}^{(1)}\rangle, \ldots, \langle \vec{x}, \vec{a}^{(m)}\rangle\right)^T. \tag{2.16}$$

Then condition (iv) means that any permutation matrix Π of order n causes a permutation $\pi \in \mathscr{S}_m$ such that

$$\Pi^* \vec{a}^{(j)} = \vec{a}^{(\pi_j)}, \qquad j = 1, \ldots, m. \tag{2.17}$$

Let us derive (2.17) from (iii) by induction on the dimension m. The case $m = 1$ is trivial because condition (iii) means that

$$\langle \vec{x}, \vec{a}^{(1)}\rangle = \langle \Pi\vec{x}, \vec{a}^{(1)}\rangle \qquad \text{for all } \vec{x},$$

or equivalently $\Pi^*\vec{a}^{(1)} = \vec{a}^{(1)}$. Suppose that the implication (iii) $\Rightarrow$ (2.17) is true for all the cases of dimension less than m. To prove (2.17) for m, we may assume that $\vec{a}^{(1)}$ has maximum norm:

$$\|\vec{a}^{(1)}\| \geqslant \|\vec{a}^{(j)}\|, \qquad j = 2, \ldots, m. \tag{2.18}$$

First claim that for any permutation matrix Π there exists $\vec{a}^{(k)}$ such that $\Pi^*\vec{a}^{(1)} = \vec{a}^{(k)}$. Apply (iii) with $\vec{x} = \vec{a}^{(1)}$ and $\vec{y} = \Pi^*\vec{a}^{(1)}$ to find a permutation $\pi \in \mathscr{S}_m$ such that

$$\begin{aligned}&\left(\langle \Pi^*\vec{a}^{(1)}, \vec{a}^{(1)}\rangle, \ldots, \langle \Pi^*\vec{a}^{(1)}, \vec{a}^{(m)}\rangle\right)^T \\ &\quad = \left(\langle \vec{a}^{(1)}, \vec{a}^{(\pi_1)}\rangle, \ldots, \langle \vec{a}^{(1)}, \vec{a}^{(\pi_m)}\rangle\right)^T.\end{aligned}$$

Take k with $\pi_k = 1$. Then

$$\langle \vec{a}^{(1)}, \Pi\vec{a}^{(k)}\rangle = \langle \Pi^*\vec{a}^{(1)}, \vec{a}^{(k)}\rangle = \langle \vec{a}^{(1)}, \vec{a}^{(1)}\rangle. \tag{2.19}$$

Since by (2.18)

$$\|\Pi\vec{a}^{(k)}\| = \|\vec{a}^{(k)}\| \leqslant \|\vec{a}^{(1)}\|,$$

according to the Cauchy-Schwarz inequality (2.19) is possible only when $\Pi\vec{a}^{(k)} = \vec{a}^{(1)}$, or equivalently $\Pi^*\vec{a}^{(1)} = \vec{a}^{(k)}$. The claim being established, there exists a subset $\mathscr{A}$ of $\{\vec{a}^{(j)} : j = 1,\dots,m\}$ such that $\vec{a}^{(1)} \in \mathscr{A}$ and $\Pi^*(\mathscr{A}) \subset \mathscr{A}$ for all permutation matrices Π. Let $\mathscr{A}_0$ be the minimum, that is, the intersection of all such $\mathscr{A}$'s. We may assume that for some $1 \leqslant m_1 \leqslant m$

$$\mathscr{A}_0 = \left\{\vec{a}^{(j)} : j = 1,\dots,m_1\right\}.$$

The requirement of minimumity indicates that each Π^* causes a bijection of $\mathscr{A}_0$. Then it follows from (iii) that the linear map $\hat{A}$ from $\mathbf{R}^n$ to $\mathbf{R}^{m-m_1}$ defined by

$$\hat{A}\vec{x} = \left(\langle\vec{x}, \vec{a}^{(m_1+1)}\rangle,\dots,\langle\vec{x}, \vec{a}^{(m)}\rangle\right)^T$$

satisfies (iii), and according to the induction assumption each Π^* causes a bijection of $\{\vec{a}^{(j)} : j = m_1 + 1,\dots,m\}$. This completes the induction. ■

COROLLARY 2.7. *Any isotone linear map A from $\mathbf{R}^n$ to itself has one of the following forms*:

(a) $A\vec{x} = \mathrm{tr}(\vec{x})\vec{a}$ *for some* $\vec{a} \in \mathbf{R}^n$.
(b) $A\vec{x} = \alpha\Pi\vec{x} + \beta\,\mathrm{tr}(\vec{x})\vec{e}$ *for some* $\alpha, \beta \in \mathbf{R}$ *and permutation* Π.

Proof. Let us use the representation (2.16). If $\vec{a}^{(1)}$ (or any $\vec{a}^{(k)}$) is not a scalar multiple of $\vec{e}$, but if all but one component of $\vec{a}^{(1)}$ (or any $\vec{a}^{(k)}$) are same, the orbits $\{\Pi^*\vec{a}^{(1)}\}$ contain exactly n distinct members when Π ranges over all permutation matrices. According to (2.17) the orbits coincide with $\{\vec{a}^{(1)},\dots,\vec{a}^{(n)}\}$. In this case A has the form (b). If all the components of $\vec{a}^{(1)}$ (or any $a^{(k)}$) are same, A has the form (a). No other case can occur. For, otherwise, the orbits $\{\Pi^*a^{(1)}\}$ would contain more than n distinct members, which contradicts (2.17). ■

The proof of Corollary 2.7 shows that any isotone linear map A form $\mathbf{R}^n$ to $\mathbf{R}^m$ with $1 \leqslant m < n$ has the form (a) for some $\vec{a} \in \mathbf{R}^m$. In particular, any isotone linear function, i.e. $m = 1$, is a scalar multiple of the trace.

Our next task is to study compatibility of definite algebraic operations with majorization. Already some unary operations were observed in (2.8) to

(2.10). To give a unified treatment for basic binary operations, let us introduce a notion. A real-valued function $\phi(s,t)$ defined on $\mathbf{R}^2$ or $\mathbf{R}^2_+$ is said to be *lattice-superadditive* if

$$\phi(s_1,t_1)+\phi(s_2,t_2)\leqslant\phi(s_1\vee s_2,t_1\vee t_2)+\phi(s_1\wedge s_2,t_1\wedge t_2).$$

Here recall that $s_1\vee s_2=\max(s_1,s_2)$ and $s_1\wedge s_2=\min(s_1,s_2)$. ϕ is said to be *monotone* if it is either monotone increasing in each argument or monotone decreasing in each argument. Each function $\phi(s,t)$ induces a map $\phi(\vec{x},\vec{y})$ from (a subset of) $\mathbf{R}^n\times\mathbf{R}^n$ to $\mathbf{R}^n$ by

$$\phi(\vec{x},\vec{y}):=\left(\phi(x_1,y_1),\dots,\phi(x_n,y_n)\right)^T.$$

THEOREM 2.8. *If $\phi(s,t)$ is monotone and lattice-superadditive, then for any $\vec{x},\vec{y}\in\mathbf{R}^n$*

$$\phi(\vec{x}^{\cdot},\vec{y}_{\cdot})\underset{\cdot}{\prec}\phi(\vec{x},\vec{y})\underset{\cdot}{\prec}\phi(\vec{x}^{\cdot},\vec{y}^{\cdot}). \tag{2.20}$$

Proof. As in the proof of Theorem 1.3, it can be shown that there exist a finite number of vectors $\vec{z}^{(0)},\dots,\vec{z}^{(N)}\in\mathbf{R}^n$ such that

$$\vec{y}^{\cdot}=\vec{z}^{(0)}\sim\cdots\sim\vec{z}^{(j)}=\vec{y}\sim\vec{z}^{(j+1)}\sim\cdots\sim\vec{z}^{(N)}=\vec{y}_{\cdot}$$

and such that, for each k, $\vec{z}^{(k+1)}$ is obtained from $\vec{z}^{(k)}$ by interchanging two components, $z_i^{(k)}$ and $z_j^{(k)}$ say, such that

$$i<j\quad\text{and}\quad z_i^{(k)}>z_j^{(k)}. \tag{2.21}$$

Therefore to prove (2.20), it suffices to show that

$$\phi(\vec{x},\vec{z}^{(k+1)})\underset{\cdot}{\prec}\phi(\vec{x},\vec{z}^{(k)}), \tag{2.22}$$

under the assumption $\vec{x}=\vec{x}^{\cdot}$, that is,

$$x_1\geqslant x_2\geqslant\cdots\geqslant x_n. \tag{2.23}$$

In view of the definition of $\phi(\vec{x},\vec{y})$ and the assumptions (2.21) and (2.23), the majorization (2.22) will follow from the following two-dimensional majoriza-

tion: if $s_1 \geqslant s_2$ and $t_1 \geqslant t_2$ then

$$(\phi(s_1,t_2),\phi(s_2,t_1))^T \underset{\cdot}{\prec} (\phi(s_1,t_1),\phi(s_2,t_2))^T. \tag{2.24}$$

But (2.24) is equivalent to saying that

$$\phi(s_1,t_2) \vee \phi(s_2,t_1) \leqslant \phi(s_1,t_1) \vee \phi(s_2,t_2) \tag{2.25}$$

and

$$\phi(s_1,t_2) + \phi(s_2,t_1) \leqslant \phi(s_1,t_1) + \phi(s_2,t_2). \tag{2.26}$$

Now (2.25) follows from the monotony of ϕ, while (2.26) follows from the lattice superadditivity. ■

Examples of monotone, lattice-superadditive functions are

$$\phi(s,t) = s+t \qquad \text{on } \mathbf{R}^2, \tag{2.27}$$

$$\phi(s,t) = s \wedge t \qquad \text{on } \mathbf{R}^2, \tag{2.28}$$

$$\phi(s,t) = st \qquad \text{on } \mathbf{R}^2_+. \tag{2.29}$$

Then Theorem 2.8 produces the following majorization relations:

$$\vec{x}^{\cdot} + \vec{y}_{\cdot} \prec \vec{x} + \vec{y} \prec \vec{x}^{\cdot} + \vec{y}^{\cdot} \qquad \text{for} \quad \vec{x},\vec{y} \in \mathbf{R}^n, \tag{2.30}$$

$$\vec{x}^{\cdot} \cdot \vec{y}_{\cdot} \underset{\cdot}{\prec} \vec{x} \cdot \vec{y} \underset{\cdot}{\prec} \vec{x}^{\cdot} \cdot \vec{y}^{\cdot} \qquad \text{for} \quad \vec{x},\vec{y} \in \mathbf{R}^n_+, \tag{2.31}$$

$$\vec{x}^{\cdot} \wedge \vec{y}_{\cdot} \underset{\cdot}{\prec} \vec{x} \wedge \vec{y} \underset{\cdot}{\prec} \vec{x}^{\cdot} \wedge \vec{y}^{\cdot} \qquad \text{for} \quad \vec{x},\vec{y} \in \mathbf{R}^n, \tag{2.32}$$

$$\vec{x}^{\cdot} \vee \vec{y}_{\cdot} \overset{\cdot}{\prec} \vec{x} \vee \vec{y} \overset{\cdot}{\prec} \vec{x}^{\cdot} \vee \vec{y}^{\cdot} \qquad \text{for} \quad \vec{x},\vec{y} \in \mathbf{R}^n. \tag{2.33}$$

Corollary 2.9. *For any $\vec{x},\vec{y} \in \mathbf{R}^n$*

$$\langle \vec{x}^{\cdot}, \vec{y}_{\cdot} \rangle \leqslant \langle \vec{x}, \vec{y} \rangle \leqslant \langle \vec{x}^{\cdot}, \vec{y}^{\cdot} \rangle. \tag{2.34}$$

Proof. When $\vec{x},\vec{y} \geqslant 0$, (2.34) results from (2.31). For general $\vec{x},\vec{y}$, take t so large that $\vec{x} + t\vec{e} \geqslant 0$ and $\vec{y} + t\vec{e} \geqslant 0$, and apply what has been just established. ■

The following lemma is useful for constructing new monotone, lattice-superadditive functions from old ones.

LEMMA 2.10.

(a) *If $f(t)$ is monotone increasing and convex, and if $\phi(s,t)$ is monotone increasing and lattice-superadditive, then the function*

$$\psi(s,t) := f(\phi(s,t))$$

is monotone and lattice-superadditive. In the case of $\phi(s,t) = s+t$, increasingness of $f(t)$ can be replaced by decreasingness.

(b) *If $f(t)$ is monotone increasing or decreasing, and if $\phi(s,t)$ is monotone and lattice-superadditive, the function*

$$\psi(s,t) := \phi(f(s), f(t))$$

is monotone and lattice-superadditive,

Proof. (a): The monotony of ψ follows from those of ϕ and f. The convexity of f and the monotone increasingness of ϕ imply

$$\begin{aligned}
&\psi(s_1 \vee s_2, t_1 \vee t_2) - \psi(s_1, t_1) \\
&\quad \geqslant f(\{\phi(s_1 \vee s_2, t_1 \vee t_2) - \phi(s_1, t_1)\} + \phi(s_1 \wedge s_2, t_1 \wedge t_2)) \\
&\qquad - f(\phi(s_1 \wedge s_2, t_1 \wedge t_2)) \\
&\quad \geqslant f(\phi(s_2, t_2)) - f(\phi(s_1 \wedge s_2, t_1 \wedge t_2))
\end{aligned}$$

because (2.20) is valid for ϕ and f is monotone increasing; and in the case of $\phi(s,t) = s+t$ increasingness of f is not necessary.

(b) is immediate, because f preserves $\vee$ and $\wedge$, or converts $\vee$ ($\wedge$) to $\wedge$ ($\vee$). ■

THEOREM 2.11. *Let $\vec{x}, \vec{y}$ be positive vectors in $\mathbf{R}^n$.*

(a) $\log(\vec{x}^{\cdot} + \vec{y}_{\cdot}) \overset{\cdot}{\prec} \log(\vec{x} + \vec{y}) \overset{\cdot}{\prec} \log(\vec{x}^{\cdot} + \vec{y}^{\cdot})$.

(b) $\log(\vec{x}^{\cdot} \cdot \vec{y}_{\cdot}) \underset{\cdot}{\prec} \log(\vec{x} \cdot \vec{y}) \underset{\cdot}{\prec} \log(\vec{x}^{\cdot} \cdot \vec{y}^{\cdot})$.

(c) $\log(\vec{x}^{\cdot} \wedge \vec{y}_{\cdot}) \underset{\cdot}{\prec} \log(\vec{x} \wedge \vec{y}) \underset{\cdot}{\prec} \log(\vec{x}^{\cdot} \wedge \vec{y}^{\cdot})$.

(c′) $\log(\vec{x}^{\cdot} \vee \vec{y}_{\cdot}) \overset{\cdot}{\prec} \log(\vec{x} \vee \vec{y}) \overset{\cdot}{\prec} \log(\vec{x}^{\cdot} \vee \vec{y}^{\cdot})$.

Proof. In view of Lemma 2.10, $-\log(s+t)$, $\log(st)$, $\log(s\wedge t)$, and $-\log(s\vee t)$ are monotone and lattice-superadditive, and Theorem 2.8 can be applied. ∎

Remark that the majorizations in Theorem 2.11 are sharper than those given in (2.30)–(2.33). For instance, given $\vec{x}$, $\vec{y}\in\mathbf{R}^n$, apply (b) of Theorem 2.11 to $\exp(\vec{x})$ and $\exp(\vec{y})$ to get (2.30).

Taking the trace in Theorem 2.11 produces the following rearrangement inequalities for $\vec{x}$, $\vec{y}\geqslant 0$:

$$\prod_{j=1}^{n}\left(x_j^{\cdot}+y_j^{\cdot}\right)\leqslant\prod_{j=1}^{n}\left(x_j+y_j\right)\leqslant\prod_{j=1}^{n}\left(x_j^{\cdot}+y_{\cdot j}\right),\tag{2.35}$$

$$\prod_{j=1}^{n}\left(x_j^{\cdot}\wedge y_{\cdot j}\right)\leqslant\prod_{j=1}^{n}\left(x_j\wedge y_j\right)\leqslant\prod_{j=1}^{n}\left(x_j^{\cdot}\wedge y_j^{\cdot}\right),\tag{2.36}$$

$$\prod_{j=1}^{n}\left(x_j^{\cdot}\vee y_j^{\cdot}\right)\leqslant\prod_{j=1}^{n}\left(x_j\vee y_j\right)\leqslant\prod_{j=1}^{n}\left(x_j^{\cdot}\vee y_{\cdot j}\right).\tag{2.37}$$

Note. The importance of convexity in connection with majorization (Corollary 2.2) was recognized by Hardy, Littlewood, and Pólya [34]. For the ordering of probability measures in terms of convex functions and the so-called Choquet theory, see Alfsen [4]. The fundamental result (Theorem 2.5) is due to Schur [71] and Ostrowski [62]. Schur [71] discovered the antiisotony of elementary symmetric functions on the set of positive vectors and found a new approach to the Hadamard determinant theorem. Various examples showing the usefulness of isotony are found in [51] and [58]. Theorem 2.6 is due to Chong [14], while Theorem 2.8 and its relatives are due to Day [18]. Lattice superadditivity is discussed in [47]. Corollary 2.9 is in Hardy, Littlewood, and Pólya [35]. For Theorem 2.11 and related inequalities, see Day [18], London [46], Minc [52], Mirsky [54], and Rudermann [67].

Most of the notions and the results of Part I can be generalized to measurable functions. See [17] for exposition. The notion of majorization plays an important role in the interpolation theory of linear operators (see the monograph [42]). There is a multidimensional generalization of majorization which produces many integral inequalities: see [10] and [49]. Applications of majorization to physics can be found in the monograph [31], and to statistics in [51] and [79].

II. DOUBLY STOCHASTIC MATRICES

3. *Double Sub- and Superstochasticity*

Let us begin with a characterization of double stochasticity in terms of majorization.

THEOREM 3.1. *The following conditions for an n-square real matrix A are mutually equivalent*:

(i) *A is doubly stochastic.*

(ii) *$A\vec{e} = \vec{e}$ and* $\operatorname{tr}|A\vec{x}| \leqslant \operatorname{tr}|\vec{x}|$ *for all $\vec{x} \in \mathbf{R}^n$.*

(iii) *$A\vec{x} \prec \vec{x}$ for all $\vec{x} \in \mathbf{R}^n$.*

(iv) *$A\vec{x} \prec \vec{x}$ for all $0 \leqslant \vec{x} \in \mathbf{R}^n$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follows from Corollary 1.2 and Theorem 1.3, and (iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i): Since $\vec{z} \prec \vec{x}$ implies $x^{\cdot}_n \leqslant z_j \leqslant x^{\cdot}_1$ for all j, $A\vec{x} \prec \vec{x}$ and $\vec{x} \geqslant 0$ yield $A\vec{x} \geqslant 0$, that is, A is positivity-preserving. Further, $A\vec{e} \prec \vec{e}$ is possible only if $A\vec{e} = \vec{e}$, that is, A is unital. To see that A is trace-preserving, for $\vec{x} \in \mathbf{R}^n$, take $t > 0$ such that $\vec{x} + t\vec{e} \geqslant 0$. Then

$$A\vec{x} + t\vec{e} = A(\vec{x} + t\vec{e}) \prec \vec{x} + t\vec{e}$$

implies $\operatorname{tr}(A\vec{x}) = \operatorname{tr}(\vec{x})$. ■

Our next aim is to establish corresponding characterizations for a doubly sub- or superstochastic matrix. Incidentally remark that (iii) in Theorem 3.1 can be replaced by each of the following:

(iii′) $A\vec{x} \underset{\cdot}{\prec} \vec{x}$ for all $\vec{x} \in \mathbf{R}^n$.

(iii″) $A\vec{x} \overset{\cdot}{\prec} \vec{x}$ for all $\vec{x} \in \mathbf{R}^n$.

For any subsets I, J of $\mathbf{N} = \{1, \ldots, n\}$ and any n-square matrix $A = (a_{ij})$, let us write

$$A(I, J) := \langle A\vec{e}_J, \vec{e}_I \rangle = \sum_{i \in I} \sum_{j \in J} a_{ij}. \tag{3.1}$$

$A(I, J)$ is defined to be 0 if one of I and J is empty.

THEOREM 3.2. *Suppose that n-square matrices $B=(b_{ij})$ and $C=(c_{ij})$ satisfy*

$$b_{ij} \geqslant c_{ij} \geqslant 0 \qquad \textit{for all } i \textit{ and } j. \tag{3.2}$$

Then there exists a doubly stochastic matrix $A=(a_{ij})$ such that

$$b_{ij} \geqslant a_{ij} \geqslant c_{ij} \qquad \textit{for all } i \textit{ and } j, \tag{3.3}$$

if and only if

$$B(I,J) \geqslant C(I^c,J^c)+|I|+|J|-n \qquad \textit{for all } I \textit{ and } J, \tag{3.4}$$

where I^c denotes the complement $\mathbf{N}\setminus I$, and $|I|$ the number of elements in I.

A proof will be given later. Immediate consequences are intrinsic characterizations of double sub- or superstochasticity of a matrix.

COROLLARY 3.3. *The following conditions for an n-square matrix $C=(c_{ij})$ with non-negative entries are mutually equivalent:*

(i) *C is doubly substochastic.*
(ii) *$C\vec{x} \underset{\cdot}{\prec} \vec{x}$ for all $0 \leqslant \vec{x} \in \mathbf{R}^n$.*
(iii) *$C\vec{e} \leqslant \vec{e}$, and $\mathrm{tr}(C\vec{x}) \leqslant \mathrm{tr}(\vec{x})$ for all $0 \leqslant \vec{x} \in \mathbf{R}^n$.*
(iv) *$C\vec{e} \leqslant \vec{e}$ and $C^*\vec{e} \leqslant \vec{e}$; in other words*

$$\sum_{j=1}^{n} c_{ij} \leqslant 1 \qquad \textit{for all } i, \tag{3.5}$$

$$\sum_{i=1}^{n} c_{ij} \leqslant 1 \qquad \textit{for all } j. \tag{3.6}$$

Proof. (i) $\Rightarrow$ (ii): By definition there exists a doubly stochastic matrix $A=(a_{ij})$ such that

$$a_{ij} \geqslant c_{ij} \qquad \text{for all } i \text{ and } j. \tag{3.7}$$

Then for any $0 \leqslant \vec{x} \in \mathbf{R}^n$

$$C\vec{x} \leqslant A\vec{x} \prec \vec{x},$$

which implies $C\vec{x} \underset{\cdot}{\prec} \vec{x}$.

(ii) $\Rightarrow$ (iii): $C\vec{e} \prec \vec{e}$ implies $C\vec{e} \leqslant \vec{e}$. $C\vec{x} \prec \vec{x}$ implies $\operatorname{tr}(C\vec{x}) \leqslant \operatorname{tr}(\vec{x})$.
(iii) $\Rightarrow$ (iv) is trivial.
(iv) $\Rightarrow$ (i): Let B be a matrix all of whose entries are equal to 1. Then the condition (3.4) becomes

$$|I|\cdot|J| - |I| - |J| + n \geqslant C(I^c, J^c). \tag{3.8}$$

But (3.5) and (3.6) imply

$$\begin{aligned} C(I^c, J^c) &\leqslant (n - |I|) \wedge (n - |J|) \\ &\leqslant |I|\cdot|J| - |I| - |J| + n. \end{aligned}$$

Therefore Theorem 3.2 guarantees the existence of a doubly stochastic matrix satisfying (3.7). ∎

Corollary 3.4. *The following conditions for an n-square matrix* $B = (b_{ij})$ *are mutually equivalent*:

(i) *B is doubly superstochastic.*
(ii) *$B\vec{x} \prec \vec{x}$ for all $0 \leqslant \vec{x} \in \mathbf{R}^n$.*
(iii) *$B(I, J) \geqslant (|I| + |J| - n)^+$ for all I and J.*

Proof. (i) $\Rightarrow$ (ii) is just as in Corollary 3.3.
(ii) $\Rightarrow$ (iii): Since $B\vec{e}_J \prec \vec{e}_J$,

$$\begin{aligned} B(I, J) = \langle B\vec{e}_J, \vec{e}_I \rangle &\geqslant \sum_{i=1}^{|I|} (B\vec{e}_J)_{\cdot i} \\ &\geqslant \sum_{i=1}^{|I|} (\vec{e}_J)_{\cdot i} = (|I| + |J| - n)^+. \end{aligned}$$

(iii) $\Rightarrow$ (i): Taking singletons for I, J, all entries of B are seen to be nonnegative. With $C = 0$ the condition (3.4) becomes

$$B(I, J) \geqslant |I| + |J| - n,$$

so that Theorem 3.2 guarantees the existence of a doubly stochastic matrix $A = (a_{ij})$ such that $b_{ij} \geqslant a_{ij}$ for all i and j. ∎

A proof of Theorem 3.2 is based on the following lemma, for the statement of which we need some generalization of the notation (3.1).

When $D=(d_{ij})$ is an $m\times n$ (rectangular) matrix for any $I\subset \mathbf{M}=\{1,\dots,m\}$ and $J\subset \mathbf{N}=\{1,\dots,n\}$, let us write

$$D(I,J):=\sum_{i\in I}\sum_{j\in J} d_{ij}. \tag{3.9}$$

$D(I,J)$ is defined equal to 0 if one of I and J is empty. For vector $\vec{x}=(x_1,\dots,x_m)^T\in\mathbf{R}^m$ and $\vec{y}=(y_1,\dots,y_n)^T\in\mathbf{R}^n$, let us use the notation

$$x(I):=\sum_{i\in I} x_i \quad\text{and}\quad y(J):=\sum_{i\in J} y_j. \tag{3.10}$$

LEMMA 3.5. *Let $D=(d_{ij})$ be an $m\times n$ matrix with nonnegative entries, and $\vec{x}=(x_1,\dots,x_m)^T\in\mathbf{R}^m$ and $\vec{y}=(y_1,\dots,y_n)^T\in\mathbf{R}^n$. Then there exists an $m\times n$ matrix $G=(g_{ij})$ such that*

$$d_{ij}\geqslant g_{ij}\geqslant 0 \qquad \textit{for all } i \textit{ and } j, \tag{3.11}$$

$$x(I)=G(I,\mathbf{N}) \;\; \textit{and} \;\; y(J)=G(\mathbf{M},J) \qquad \textit{for all } I \textit{ and } J, \tag{3.12}$$

if and only if the following conditions are satisfied:

$$x(\mathbf{M})=y(\mathbf{N}), \tag{3.13}$$

and

$$x(I)+y(J)\leqslant D(I,J)+y(\mathbf{N}) \qquad \textit{for all } I \textit{ and } J. \tag{3.14}$$

Remark that the pair of conditions (3.13) and (3.14) is equivalently expressed by the following pair of conditions:

$$x(I)\leqslant D(I,J)+y(J^c) \qquad \text{for all } I \text{ and } J, \tag{3.15}$$

$$y(J)\leqslant D(I,J)+x(I^c) \qquad \text{for all } I \text{ and } J. \tag{3.16}$$

Postponing a proof of Lemma 3.5, let us first derive Theorem 3.2 from Lemma 3.5.

Suppose that B and C satisfy (3.2), and let

$$D:=B-C, \qquad \vec{x}:=\vec{e}-C\vec{e}, \quad\text{and}\quad \vec{y}:=\vec{e}-C^*\vec{e}.$$

Then there exists a doubly stochastic matrix A satisfying (3.3) if and only if there exists a matrix G satisfying (3.11) and (3.12). (Here $m = n$ and $\mathbf{M} = \mathbf{N}$). In fact, G is related to A by $G = A - C$. Since

$$x(\mathbf{N}) = n - C(\mathbf{N},\mathbf{N}) = y(\mathbf{N}),$$

according to Lemma 3.5 the condition for the existence of G is expressed by

$$|I| - C(I,\mathbf{N}) + |J| - C(\mathbf{N}, J) \leqslant B(I, J) - C(I, J) + n - C(\mathbf{N},\mathbf{N}),$$

which reduces to (3.4).

Proof of Lemma 3.5. Suppose first that (3.11) and (3.12) are fulfilled. Then

$$x(\mathbf{M}) = G(\mathbf{M},\mathbf{N}) = y(\mathbf{N}),$$

and

$$\begin{aligned} x(I) = G(I,\mathbf{N}) &\leqslant G(I, J) + G(\mathbf{M}, J^c) \\ &\leqslant D(I, J) + y(J^c) \\ &= D(I, J) + y(\mathbf{N}) - y(J). \end{aligned}$$

Therefore (3.13) and (3.14) are satisfied.

Suppose conversely that (3.13) and (3.14), or equivalently (3.15) and (3.16), are fulfilled. We proceed by induction on the sum $m + n$. When $m + n = 2$, that is, $m = n = 1$, the assumption means

$$0 \leqslant x_1 = y_1 \leqslant d_{11},$$

so that $g_{11} := x_1$ meets the requirement. Suppose that the implication $(3.15)\&(3.16) \Rightarrow (3.11)\&(3.12)$ is true for all the cases of dimensions m', n' with $m' + n' < m + n$. Define, for $0 \leqslant t \leqslant 1$, a function $H^{(t)}(I, J)$ for all $I \subset \mathbf{M} = \{1,\ldots,m\}$ and $J \subset \mathbf{N} = \{1,\ldots,n\}$ by

$$\begin{aligned} H^{(t)}(I, J) := tD(I, J) - x(I) + y(J^c) \\ = tD(I, J) - y(J) + x(I^c). \end{aligned} \tag{3.17}$$

Since $H^{(1)}(I,J) \geqslant 0$ by (3.15), let

$$t_0 = \min\{ t : H^{(t)}(I,J) \geqslant 0 \text{ for all } I \text{ and } J \}. \tag{3.18}$$

If $t_0 = 0$, then $\vec{x} = 0$ and $\vec{y} = 0$, and $G := 0$ meets the requirement. If $t_0 > 0$, according to the minimum property (3.18) there exist $I_0 \subset \mathbf{M}$ and $J_0 \subset \mathbf{N}$ such that

$$H^{(t_0)}(I_0, J_0) = 0, \quad \text{but} \quad H^{(t)}(I_0, J_0) < 0 \quad \text{for all } t < t_0. \tag{3.19}$$

The case that one of I_0 and J_0 is empty is excluded, because $H^{(t)}(I_0, J_0)$ becomes independent of t, contradicting (3.19).

Define a $|I_0| \times |J_0^c|$ matrix G' and vectors $\vec{x}' \in \mathbf{R}^{|I_0|}$ and $\vec{y}' \in \mathbf{R}^{|J_0^c|}$ by the relations

$$x'(I) := x(I) - t_0 D(I, J_0) \qquad \text{for all} \quad I \subset I_0, \tag{3.20}$$

$$y'(J) := y(J) \qquad \text{for all} \quad J \subset J_0^c, \tag{3.21}$$

and

$$D'(I,J) := t_0 D(I,J) \qquad \text{for all} \quad I \subset I_0 \text{ and } J \subset J_0^c. \tag{3.22}$$

Then (3.21) implies

$$x'(I_0) = y'(J_0^c).$$

Since by (3.18) $H^{(t_0)}(I,J) \geqslant 0$ for all I and J,

$$\begin{aligned} x'(I) &= x(I) - t_0 D(I, J_0) \\ &\leqslant t_0 D(I, J_0 \cup J) + y(J_0^c \setminus J) - t_0 D(I, J_0) \\ &= D'(I,J) + y(J_0^c \setminus J) \qquad \text{for all} \quad I \subset I_0 \text{ and } J \subset J_0^c. \end{aligned}$$

These show that the triple $(D', \vec{x}', \vec{y}')$ with I_0 and J_0^c in place of $\mathbf{M}$ and $\mathbf{N}$ respectively satisfies (3.15) and (3.16). Since $|I_0| + |J_0^c| < m + n$, according to the induction assumption there exists a matrix $G' = (g'_{ij})_{i \in I_0,\, j \in J_0^c}$ such that

$$t_0 d_{ij} = d'_{ij} \geqslant g'_{ij} \geqslant 0 \qquad \text{for all} \quad i \in I_0 \text{ and } j \in I_0^c, \tag{3.23}$$

and

$$G'(I, J_0^c) = x(I) - t_0 D(I, J_0) \quad \text{and} \quad G'(I_0, J) = y(J)$$
$$\text{for} \quad I \subset I_0 \text{ and } J \subset J_0^c. \quad (3.24)$$

Similarly there exists a matrix $G'' = (g_{ij}'')_{i \in I_0^c,\ j \in J_0}$ such that

$$t_0 d_{ij} \geqslant g_{ij}'' \geqslant 0 \qquad \text{for all} \quad i \in I_0^c \text{ and } j \in J_0, \qquad (3.25)$$

and

$$G''(I, J_0) = x(I) \quad \text{and} \quad G''(I_0^c, J) = y(J) - t_0 D(I_0, J)$$
$$\text{for} \quad I \subset I_0^c \text{ and } J \subset J_0. \quad (3.26)$$

Finally define an $m \times n$ matrix $G = (g_{ij})$ by the relation

$$G(I, J) = t_0 D(I \cap I_0, J \cap J_0) + G'(I \cap I_0, J \cap J_0^c)$$
$$+ G''(I \cap I_0^c, J \cap J_0) \qquad \text{for all } I \text{ and } J.$$

Then (3.11) follows from (3.23) and (3.25), while (3.24) and (3.26) imply (3.12):

$$G(I,\mathrm{N}) = t_0 D(I \cap I_0, J_0) + G'(I \cap I_0, J_0^c) + G''(I \cap I_0^c, J_0)$$
$$= x(I \cap I_0) + x(I \cap I_0^c) = x(I)$$

and

$$G(\mathrm{M}, J) = y(J).$$

The matrix G meets the requirement. ■

NOTE. We followed Kellerer [39] in proving the key result (Lemma 3.5). Dr. Ch. Nara pointed out that Lemma 3.5 could be derived from the theory of flow in networks (see [27]). Corollary 3.3 is in von Neumann [60], while Corollary 3.4 stands as an open problem in [51].

4. *Doubly Stochastic Matrices*

Our first aim is to show the special role of permutation matrices among doubly stochastic matrices.

The *permanent* $\operatorname{per}(A)$ of an (n-square) matrix $A=(a_{ij})$ is defined by

$$\operatorname{per}(A)=\sum_{\pi\in\mathcal{S}_n} a_{1\pi_1}\cdots a_{n\pi_n}. \tag{4.1}$$

The permanent is *permutation-invariant* in the sense

$$\operatorname{per}(A)=\operatorname{per}(\Pi^{(1)}A\Pi^{(2)}) \qquad \text{for all permutations } \Pi^{(1)},\Pi^{(2)}. \tag{4.2}$$

For an (n-square) matrix $A=(a_{ij})$ and $I,\ J\subset \mathbf{N}=\{1,\dots,n\}$, let us denote by $A_{I,J}$ the small rectangular matrix

$$A_{I,J}=(a_{ij})_{i\in I,\ j\in J}. \tag{4.3}$$

An (n-square) matrix A is said to be *partly decomposable* if there exist nonempty $I,\ J\subset\mathbf{N}$ such that

$$|I|+|J|=n \quad \text{and} \quad A_{I,J}=0. \tag{4.4}$$

It is said to be *decomposable* if J in (4.4) can be taken equal to I^c. When (4.4) is fulfilled, the following holds:

$$\operatorname{per}(A)=\operatorname{per}(A_{I,J^c})\operatorname{per}(A_{I^c,J}). \tag{4.5}$$

A matrix is said to be *indecomposable* (*fully indecomposable*) if it is not decomposable (not partly decomposable).

If all entries of A are nonnegative, then for any $I,\ J\subset\mathbf{N}$ with $|I|+|J|=n$,

$$\operatorname{per}(A)\geqslant \operatorname{per}(A_{I,J^c})\operatorname{per}(A_{I^c,J}). \tag{4.6}$$

LEMMA 4.1. *Let $A=(a_{ij})$ be an n-square matrix with nonnegative entries. Then its permanent vanishes,* $\operatorname{per}(A)=0$, *if and only if there exist I, $J\subset\mathbf{N}$ such that*

$$|I|+|J|\geqslant n+1 \quad \textit{and} \quad A(I,J)=0. \tag{4.7}$$

Proof. Suppose that (4.7) is fulfilled. According to permutation invariance (4.2), we may assume that, for some k, $J=\{1,\dots,k\}$ and $I=\{k,\dots,n\}$. Then $A_{I,I^c}=0$ and the first column of $A_{I,I}$ vanishes, and $\operatorname{per}(A)=0$ follows from (4.5).

We proceed by induction to prove the converse implication. The case $n = 1$ is trivial. Suppose that the assertion is true for all the cases of dimension less than n, and take a nonzero n-square matrix A with nonnegative entries such that $\mathrm{per}(A) = 0$. We may assume $a_{nn} > 0$. Then (4.6) implies $\mathrm{per}(A_{\mathbf{N}\setminus\{n\},\mathbf{N}\setminus\{n\}}) = 0$. According to the induction assumption, there exist I', $J' \subset \{1,\dots,n-1\}$ such that

$$|I'|+|J'| \geqslant n \quad \text{and} \quad A(I',J') = 0. \tag{4.8}$$

If $|I'|+|J'| \geqslant n+1$, let $I := I'$ and $J := J'$ for (4.7). Suppose $|I'|+|J'| = n$. Since (4.6) with $\mathrm{per}(A) = 0$ implies that one of $\mathrm{per}(A_{I',J'^c})$ and $\mathrm{per}(A_{I'^c,J'})$ vanishes, we may assume $\mathrm{per}(A_{I',J'^c}) = 0$. Again according to the induction assumption, there exist $I'' \subset I'$ and $J'' \subset J'^c$ such that

$$|I''|+|J''| \geqslant |I'|+1 \quad \text{and} \quad A(I'',J'') = 0. \tag{4.9}$$

Then it follows from (4.8) and (4.9) that

$$|I''|+|J' \cup J''| = |I''|+|J''|+|J'| \geqslant n+1$$

and

$$0 \leqslant A(I'',J' \cup J'') \leqslant A(I'',J'') + A(I',J') = 0,$$

and $I := I''$ and $J := = J' \cup J''$ meet the requirement. ∎

Corollary 4.2. *Let A be an n-square matrix with nonnegative entries.*

(a) *If A is doubly stochastic, then* $\mathrm{per}(A) > 0$.

(b) *If A is fully indecomposable, then* $\mathrm{per}(A_{\langle i,j\rangle}) > 0$ *for all i and j, where $A_{\langle i,j\rangle}$ is the $(n-1)$-square matrix obtained from A by deleting the ith row and the jth column.*

Proof. (a): If $\mathrm{per}(A) = 0$, according to Lemma 4.1 there exist I and J satisfying (4.7). Then double stochasticity implies

$$\begin{aligned} 0 = A(I,J) &\geqslant A(I,\mathbf{N}) - A(\mathbf{N},J^c) \\ &= |I| - |J^c| \geqslant 1, \end{aligned}$$

which is a contradiction.

(b): If $\mathrm{per}(A_{\langle i,j\rangle})=0$, according to Lemma 4.1 there exist I and J such that $i\notin I$, $j\notin J$, $|I|+|J|\geqslant n$, and $A(I,J)=0$, contradicting the full indecomposability of A. ■

It is easy to see that each permutation matrix Π of order n is an extreme point of the convex set Ω_n of all n-square doubly stochastic matrices, in the sense that $\Pi = tA+(1-t)B$ with $A, B\in\Omega_n$, $0<t<1$ is possible only when $A=B=\Pi$. The next theorem establishes the converse.

THEOREM 4.3. *Any (n-square) doubly stochastic matrix $A=(a_{ij})$ is a convex combination of permutation matrices.*

Proof by induction on the number $\kappa(A)$ of nonzero entries of A. Since each row contains at least one nonzero entry, $\kappa(A)\geqslant n$. If $\kappa(A)=n$, then A is a permutation matrix. Suppose that any doubly stochastic matrix B with $\kappa(B)<\kappa(A)$ is a convex combination of permutation matrices. Since $\mathrm{per}(A)>0$ by Corollary 4.2, there exists a permutation $\pi\in\mathscr{S}_n$ such that $a_{j\pi_j}>0$ for all j. Let

$$t_0 := \min_{1\leqslant j\leqslant n} a_{j\pi_j} \quad \text{and} \quad \Pi^{(0)} := \left(\delta_{i\pi_j}\right).$$

If $t_0=1$, then A must coincides with $\Pi^{(0)}$. If $t_0<1$, the matrix $B:=(A-t_0\Pi^{(0)})/(1-t_0)$ is doubly stochastic with $\kappa(B)<\kappa(A)$. According to the induction assumption, B is a convex combination of permutation matrices; then so is A. ■

COROLLARY 4.4. *Any (n-square) doubly substochastic matrix $C=(c_{ij})$ is a convex combination of matrices of the form* $\mathrm{diag}(\vec{e}_I)\cdot\Pi$, *where Π is a permutation matrix and $I\subset\{1,\dots,n\}$.*

Proof. By definition there exists a doubly stochastic matrix $A=(a_{ij})$ such that $a_{ij}\geqslant c_{ij}\geqslant 0$ for all i and j. According to Theorem 4.3, there exist $t_k>0$ and permutation matrices $\Pi^{(k)}$ such that

$$\sum_{k=1}^{N} t_k = 1 \quad \text{and} \quad A=\sum_{k=1}^{N} t_k\Pi^{(k)}.$$

Then C is decomposed in the form

$$C = \sum_{k=1}^{N} t_k \operatorname{diag}(\vec{a}^{(k)}) \cdot \Pi^{(k)},$$

where $0 \leqslant \vec{a}^{(k)} \leqslant \vec{e}$. Therefore, to prove the assertion, it suffices to show that any vector $\vec{a}$ such that $0 \leqslant \vec{a} \leqslant \vec{e}$ is a convex combination of vectors of the form $\vec{e}_I$. To this end, we may assume that $\vec{a} = \vec{a}^{\bullet}$, that is,

$$0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant 1.$$

Let $I^{(j)} = \{j, j+1, \ldots, n\}$, $j = 1, \ldots, n$. Then

$$\vec{a} = a_1 \vec{e}_{I^{(1)}} + \sum_{j=2}^{n} (a_j - a_{j-1}) \vec{e}_{I^{(j)}}. \qquad \blacksquare$$

The following is a stochastic analogue of Theorem 4.3.

THEOREM 4.5. *Any stochastic matrix* $A = (a_{ij})$ *is a convex combination of* $(0,1)$*-valued stochastic matrices.*

The proof is parallel to that of Theorem 4.3 and is based on induction on the number $\kappa(A)$ of nonzero entries of A. Suppose that the assertion is true for all the cases of the number $< \kappa(A)$. For each j, find τ_j such that

$$a_{\tau_j j} = \min_i \{a_{ij} : a_{ij} > 0\},$$

and let

$$t_0 := \min_{1 \leqslant j \leqslant n} a_{\tau_j j} \quad \text{and} \quad Q^{(0)} = \left(\delta_{i\tau_j}\right).$$

$Q^{(0)}$ is a $(0,1)$-valued stochastic matrix. If $t_0 = 1$, then A is $(0,1)$-valued. If $t_0 < 1$, then the matrix $B := (A - t_0 Q^{(0)})/(1 - t_0)$ is stochastic with $\kappa(B) < \kappa(A)$. According to the induction assumption, B is a convex combination of $(0,1)$-valued stochastic matrices, and so is A. $\blacksquare$

Our next aim is to find a stochastic or doubly stochastic matrix that is intrinsically related to a given matrix with nonnegative entries.

The *support* of a vector $\vec{x} \in \mathbf{R}^n$ is the subset of $\mathbf{N}$

$$I_{\vec{x}} = \{ j : x_j \neq 0 \}.$$

A vector $\vec{x} \in \mathbf{R}^n$ is said to be *strictly positive* if $\vec{x} \geqslant 0$ and $I_{\vec{x}} = \mathbf{N}$. When A is a matrix with nonnegative entries, the relation $A\vec{x} = 0$ for a vector $\vec{x} \geqslant 0$ is equivalently expressed either by $\mathrm{tr}(A\vec{x}) = 0$ or $A(\mathbf{N}, I_{\vec{x}}) = 0$.

LEMMA 4.6. *Let A be an (n-square) matrix with nonnegative entries. If A is indecomposable, there exist a positive number $\rho > 0$ and a strictly positive vector $\vec{x}^{(0)}$ such that*

$$\|\vec{x}^{(0)}\| = 1 \quad \textit{and} \quad A\vec{x}^{(0)} = \rho \vec{x}^{(0)}. \tag{4.10}$$

ρ is an eigenvalue with maximum modulus of A, and $\vec{x}^{(0)}$ is a unique (up to a scalar) eigenvector of A corresponding to ρ.

Proof. Consider a compact convex set

$$\mathscr{A} := \{ \vec{x} \geqslant 0 : \mathrm{tr}(\vec{x}) = 1 \},$$

and define a map Φ on $\mathscr{A}$ by

$$\Phi(\vec{x}) := \frac{A^*\vec{x}}{\mathrm{tr}(A^*\vec{x})}.$$

Φ is well defined. For, if $\mathrm{tr}(A^*\vec{x}) = 0$ for some $\vec{x} \in \mathscr{A}$, then $\vec{x} \neq 0$ and $A(I_{\vec{x}}, \mathbf{N}) = 0$, contradicting the indecomposability of A. Now since Φ is a continuous map from the compact convex set into itself, according to the Brouwer fixed-point theorem there exists $\vec{y} \in \mathscr{A}$ such that $\Phi(\vec{y}) = \vec{y}$. With $\rho := \mathrm{tr}(A^*\vec{y}) > 0$ this means

$$A^*\vec{y} = \rho \vec{y}. \tag{4.11}$$

$\vec{y}$ is strictly positive. For, otherwise, (4.11) implies $A(I_{\vec{y}}, I_{\vec{y}}^c) = 0$, contradicting the indecomposability of A. Apply the same procedure to A instead A^* to find a positive number $\rho' > 0$ and a strictly positive vector $\vec{x}$ such that $A\vec{x} = \rho'\vec{x}$. If $\rho \neq \rho'$, then $\langle \vec{x}, \vec{y} \rangle = 0$, contradicting strict positivity. Therefore ρ and $\vec{x}^{(0)} := \vec{x}/\|\vec{x}\|$ satisfy (4.10). Take any (complex) vector $\vec{z}$ such that

$A\vec{z} = \rho\vec{z}$. Find a complex number α such that the support of $\vec{z} - \alpha\vec{x}^{(0)}$ does not coincide with N. Since all entries of A are nonnegative,

$$A(\vec{z} - \alpha\vec{x}^{(0)}) = \rho(\vec{z} - \alpha\vec{x}^{(0)})$$

implies

$$A|\vec{z} - \alpha\vec{x}^{(0)}| - \rho|\vec{z} - \alpha\vec{x}^{(0)}| \geqslant 0. \tag{4.12}$$

The scalar product of the left side of (4.12) with $\vec{y}$ vanishes by (4.11). Because $\vec{y}$ is strictly positive, this is possible only when

$$A|\vec{z} - \alpha\vec{x}^{(0)}| = \rho|\vec{z} - \alpha\vec{x}^{(0)}|. \tag{4.13}$$

Since the support of $\vec{z} - \alpha\vec{x}^{(0)}$ does not coincide with N, and A is indecomposable, (4.13) is possible only when $\vec{z} - \alpha\vec{x}^{(0)} = 0$. Thus $\vec{x}^{(0)}$ is a unique (up to a scalar) eigenvector of A corresponding to ρ. Finally let $\vec{w}$ be a nonzero eigenvector of A corresponding to an eigenvalue λ,

$$A\vec{w} = \lambda\vec{w}.$$

Then, as above, $A|\vec{w}| \geqslant |\lambda|\,|\vec{w}|$, and (4.11) implies $\rho \geqslant |\lambda|$. ■

An immediate consequence is the diagonal similarity of a matrix with nonnegative entries to a stochastic matrix.

THEOREM 4.7. *Let A be a square matrix with nonnegative entries. If A is indecomposable, there exist uniquely a positive number $\rho > 0$, a strictly positive vector $\vec{a}$ with $\|\vec{a}\| = 1$, and a stochastic matrix S such that*

$$\operatorname{diag}(\vec{a})\,A = \rho S \operatorname{diag}(\vec{a}). \tag{4.14}$$

Proof. Apply Lemma 4.6 to the indecomposable matrix A^* to find $\rho > 0$ and a strictly positive vector $\vec{a}$ such that

$$\|\vec{a}\| = 1 \quad \text{and} \quad A^*\vec{a} = \rho\vec{a}. \tag{4.15}$$

Let

$$S := \rho^{-1}\operatorname{diag}(\vec{a})\,A\operatorname{diag}(\vec{a})^{-1}.$$

Then S is a matrix with nonnegative entries, and (4.14) is satisfied. Since (4.15) implies $S^*\vec{e} = \vec{e}$, S is stochastic.

Suppose that $\rho' > 0$, strictly positive $\vec{a}'$ with $\|\vec{a}'\| = 1$, and a stochastic matrix S' also satisfy (4.14). Then

$$S^*(\vec{a}'/\vec{a}) = \rho'\rho^{-1}\vec{a}'/\vec{a}. \tag{4.16}$$

Since S^* is indecomposable together with A, it follows from (4.16) via Lemma 4.6 that $\rho' = \rho$ and $\vec{a}' = \vec{a}$, because $\|\vec{a}'\| = \|\vec{a}\|$, and consequently $S' = S$. ■

An indecomposable matrix with nonnegative entries is not always similar to a scalar multiple of a doubly stochastic matrix, yet the following is true.

THEOREM 4.8. *Let A be an n-square matrix with nonnegative entries. If A is fully indecomposable, then there exist uniquely a doubly stochastic matrix D, a strictly positive vector $\vec{a}$ with $\|\vec{a}\| = 1$, and a strictly positive vector $\vec{b}$ such that*

$$\mathrm{diag}(\vec{a})\, A = D\,\mathrm{diag}(\vec{b}). \tag{4.17}$$

Proof. Suppose first that all entries of A are positive. As in the proof of Lemma 4.6, using the Brouwer fixed-point theorem, find $\rho > 0$ and strictly positive $\vec{a}$ with $\|\vec{a}\| = 1$ such that

$$\left[A(A^*\vec{a})^{-1}\right]^{-1} = \rho\vec{a}. \tag{4.18}$$

Let

$$\vec{b} := A^*\vec{a} \quad \text{and} \quad D := \mathrm{diag}(\vec{a})\, A\,\mathrm{diag}(\vec{b})^{-1}. \tag{4.19}$$

Then $\vec{b}$ is strictly positive and all entries of D are positive. Since $\vec{b}^{-1}\cdot A^*\vec{a} = \vec{e}$ by definition, $D^*\vec{e} = \vec{e}$. On the other hand, (4.18) implies

$$\rho\vec{a}\cdot A(\vec{b}^{-1}) = \vec{e}$$

or equivalently

$$\rho D\vec{e} = \vec{e}.$$

Then by Lemma 4.6 $\rho = 1$, and consequently D is doubly stochastic. The relation (4.17) follows from (4.19).

Turning to the case of fully indecomposable A, take a sequence of matrices $A^{(k)}$, all of whose entries are positive, and which converges to A. It follows from what has been just proved that there exist doubly stochastic matrices $D^{(k)}$, $\vec{a}^{(k)} \geqslant 0$ with $\|\vec{a}^{(k)}\| = 1$, and $\vec{b}^{(k)} \geqslant 0$ such that

$$\operatorname{diag}(\vec{a}^{(k)})\, A^{(k)} = D^{(k)} \operatorname{diag}(\vec{b}^{(k)}), \qquad k = 1, 2, \ldots .$$

We may assume that $\vec{a}^{(k)}$ converges to $\vec{a} \geqslant 0$ with $\|\vec{a}\| = 1$, and $D^{(k)}$ converges to a doubly stochastic matrix D. Even $\vec{b}^{(k)}$ can be assumed to converge to $\vec{b} \geqslant 0$, because $\vec{b}^{(k)} = A^{(k)*}\vec{a}^{(k)}$. For D, $\vec{a}$, and $\vec{b}$ the relation (4.17) holds. Then (4.17) implies

$$A(I_{\vec{a}}, I_{\vec{b}}^c) = 0 \quad \text{and} \quad D(I_{\vec{a}}^c, I_{\vec{b}}) = 0. \tag{4.20}$$

(Recall that $I_{\vec{a}}$ and $I_{\vec{b}}$ are the supports of $\vec{a}$ and $\vec{b}$ respectively.) Since $\vec{a} \neq 0$ and A is fully indecomposable, the first relation of (4.20) implies that if $I_{\vec{b}}^c \neq \varnothing$

$$|I_{\vec{a}}| + |I_{\vec{b}}^c| < n. \tag{4.21}$$

On the other hand, since $\operatorname{per}(D) > 0$ by Corollary 4.2, the second relation of (4.20) implies, via Lemma 4.1,

$$|I_{\vec{a}}^c| + |I_{\vec{b}}| \leqslant n. \tag{4.22}$$

Since $|I_{\vec{a}}| > 0$, (4.21) and (4.22) are consistent only when $I_{\vec{a}} = I_{\vec{b}} = \mathbf{N}$, that is, $\vec{a}$ and $\vec{b}$ are strictly positive.

The proof of uniqueness is quite similar to that in Theorem 4.7, and is omitted. ■

NOTE. Lemma 4.1 is called the Frobenius-König theorem [29]. The fundamental result (Theorem 4.3) is due to Birkhoff [9]. A survey of the research on doubly stochastic matrices with some open problems can be found in [57]. Lemma 4.6 and Theorem 4.7 are only a small part of the so-called Perron-Frobenius theorem [63, 28]. More about indecomposable matrices can be found in [83] and the monographs Gantmacher [30] and Seneta [72]. Theorem 4.8 was first proved by Sinkhorn [73] for a matrix with

positive entries. Subsequent generalizations are due to Brualdi, Parter and Schneider [11], and Sinkhorn and Knopp [74].

5. *Doubly Stochastic Matrix with Minimum Permanent*

In Corollary 4.2 it was shown that per(A) is positive on the set Ω_n of doubly stochastic matrices of order n. Let

$$\gamma_n := \min\{\operatorname{per}(A) : A \in \Omega_n\}. \tag{5.1}$$

Let us call A in Ω_n a *minimizing matrix* if $\operatorname{per}(A) = \gamma_n$. It has been a long-standing conjecture (the *van der Waerden conjecture*) that

$$\mathbf{J}_n = \begin{bmatrix} 1/n & \cdots & 1/n \\ \vdots & & \vdots \\ 1/n & \cdots & 1/n \end{bmatrix} \tag{5.2}$$

is a unique minimizing matrix; *a fortiori*,

$$\gamma_n = n!/n^n. \tag{5.3}$$

Very recently this conjecture was settled affirmatively by Egorychev. In this section we shall follow his line of ideas to the solution.

As the case $n = 1$ is trivial, we may proceed with the assumption that the conjecture is true for all dimensions less than n.

LEMMA 5.1. *Any minimizing matrix A is fully indecomposable.*

Proof. Suppose that $|I| + |J| = n$ and $A(I, J) = 0$. Then double stochasticity implies

$$|J| = A(\mathbf{N}, J) = A(I^c, J)$$

and

$$|J| = |I^c| = A(I^c, J^c) + A(I^c, J),$$

which yield $A(I^c, J^c) = 0$. Therefore A_{I, J^c} is a doubly stochastic matrix of order $k := |I|$, while $A_{I^c, J}$ is one of order $n - k$. Since by (4.5)

$$\operatorname{per}(A) = \operatorname{per}(A_{I, J^c}) \operatorname{per}(A_{I^c, J}),$$

it follows from the minimizing property of A and the induction assumption that

$$\frac{n!}{n^n} \geqslant \operatorname{per}(A) \geqslant \frac{k!}{k^k} \cdot \frac{(n-k)!}{(n-k)^{n-k}}.$$

But this is possible only when $k = 0$ or $k = n$, that is, $I = \varnothing$ or $J = \varnothing$. ∎

Together with A, the matrices A^*A and AA^* are fully indecomposable. In fact, suppose, for instance, that there exist nonempty I, J such that

$$|I| + |J| = n \quad \text{and} \quad (A^*A)(I, J) = 0.$$

Then $A\vec{e}_I$ and $A\vec{e}_J$ have disjoint supports. Let J_1 and I_1 denote the complement of the support of $A\vec{e}_I$ and $A\vec{e}_J$ respectively. By definition

$$A(J_1, I) = A(I_1, J) = 0,$$

and

$$|I_1| + |J_1| \geqslant n.$$

Then either $|J_1| + |I| \geqslant n$ or $|I_1| + |J| \geqslant n$, both of which contradict the full indecomposability of A.

Considering $\operatorname{per}(A)$ as a multilinear function of column vectors, let us also use the notation

$$\operatorname{per}(A) = \operatorname{per}(\vec{a}^{(1)}, \dots, \vec{a}^{(n)})$$

where $\vec{a}^{(j)}$ is the jth column vector of A.

LEMMA 5.2. *If $\vec{a}^{(j)}$, $j = 1, \dots, n-1$, are strictly positive n-vectors, then for any (complex) n-vector $\vec{x} = (x_1, \dots, x_n)^T$*

$$\left|\operatorname{per}(\vec{a}^{(1)}, \dots, \vec{a}^{(n-1)}, \vec{x})\right|^2$$
$$\geqslant \operatorname{per}(\vec{a}^{(1)}, \dots, \vec{a}^{(n-1)}, \vec{a}^{(n-1)}) \cdot \operatorname{per}(\vec{a}^{(1)}, \dots, \vec{a}^{(n-2)}, \vec{x}, \vec{\bar{x}}), \quad (5.4)$$

where $\vec{\bar{x}} = (\bar{x}_1, \dots, \bar{x}_n)^T$. Here equality occurs only when $\vec{x}$ is a scalar multiple of $\vec{a}^{(n-1)}$. The inequality (5.4) itself is valid if $\vec{a}^{(j)} \geqslant 0$, $j = 1, \dots, n-1$.

Proof by induction on the dimension n. There exists a unique Hermitian matrix S such that

$$\operatorname{per}\left(\vec{a}^{(1)},\dots,\vec{a}^{(n-2)},\vec{x},\vec{y}\right)=\langle S\vec{x},\vec{y}\rangle \qquad \text{for all} \quad \vec{x},\vec{y}\in \mathbf{C}^n. \tag{5.5}$$

Since $\langle S\vec{a}^{(n-1)},\vec{a}^{(n-1)}\rangle > 0$, the largest eigenvalue $\lambda_1^{\cdot}(S)$ of S is positive, and in view of Corollary 6.6, proved in the next section, the assertion of the lemma is equivalent to the condition that $\lambda_2^{\cdot}(S)$, the second largest eigenvalue of S, is negative. For $n = 2$, then

$$S=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

and this condition is satisfied. For general n, this condition is satisfied when $\vec{a}^{(1)}=\cdots=\vec{a}^{(n-2)}=\vec{e}$. Indeed, the largest eigenvalues is $(n-1)!$ and all other eigenvalues coincide with $-(n-2)!$. Let

$$\vec{a}^{(j)}(t):=(1-t)\vec{e}+t\vec{a}^{(j)} \qquad \text{for} \quad 0\leqslant t\leqslant 1, \quad j=1,\dots,n-2,$$

and denote by S_t the associated Hermitian matrix. Since $\lambda_2^{\cdot}(S_0)=-(n-2)!$ and $\lambda_2^{\cdot}(S_t)$ depends continuously on t, the conclusion $\lambda_2^{\cdot}(S)\equiv\lambda_2^{\cdot}(S_1)<0$ will follow if all S_t are invertible. Therefore it suffices to prove generally the invertibility of a Hermitian matrix associated with strictly positive vectors under the assumption that for any strictly positive $(n-1)$-vectors $\vec{b}^{(1)},\dots,\vec{b}^{(n-3)}$ the associated Hermitian matrix T of order $n-1$ satisfies the condition $\lambda_1^{\cdot}(T)>0>\lambda_2^{\cdot}(T)$.

Let anew S be a Hermitian matrix of order n, associated with strictly positive n-vectors $\vec{a}^{(1)},\dots,\vec{a}^{(n-2)}$. Suppose that $S\vec{x}=0$ for some nonzero vector $\vec{x}$ with $x_k\neq 0$, say. Since S is a real matrix, we may assume that $\vec{x}$ is a real vector. For any j and any n-vector $\vec{z}$, let $\vec{z}_{\langle j\rangle}$ stand for the $(n-1)$-vector obtained from $\vec{z}$ by deleting its jth component. According to the induction assumption, the Hermitian matrix $S_{\langle j\rangle}$ of order $n-1$, associated with strictly positive vectors $\vec{a}^{(1)}_{\langle j\rangle},\dots,\vec{a}^{(n-3)}_{\langle j\rangle}$, satisfies $\lambda_1^{\cdot}(S_{\langle j\rangle})>0>\lambda_2^{\cdot}(S_{\langle j\rangle})$. But since

$$\begin{aligned}0=(S\vec{x})_j&=\operatorname{per}\left(\vec{a}^{(1)}_{\langle j\rangle},\dots,\vec{a}^{(n-2)}_{\langle j\rangle},\vec{x}_{\langle j\rangle}\right)\\&=\langle S_{\langle j\rangle}\vec{a}^{(n-2)}_{\langle j\rangle},\vec{x}_{\langle j\rangle}\rangle,\end{aligned}$$

Corollary 6.6 tells us that

$$\langle S_{\langle j\rangle}\vec{x}_{\langle j\rangle}, \vec{x}_{\langle j\rangle}\rangle \leqslant 0 \tag{5.6}$$

and that if $\vec{x}_{\langle j\rangle} \neq 0$ then equality is excluded in (5.6). Since $\vec{x}_{\langle j\rangle} \neq 0$ for $j \neq k$, it follows from (5.6) that

$$0 = \langle S\vec{x}, \vec{x}\rangle = \sum_{j=1}^{n} a_j^{(n-2)}\langle S_{\langle j\rangle}\vec{x}_{\langle j\rangle}, \vec{x}_{\langle j\rangle}\rangle < 0,$$

a contradiction. The last assertion of the lemma results from a continuity argument. ∎

LEMMA 5.3. *If $A = (a_{ij})$ is a minimizing matrix, then*

$$\operatorname{per}(A_{\langle i,j\rangle}) = \operatorname{per}(A) \qquad \text{for all } i \text{ and } j. \tag{5.7}$$

Here $A_{\langle i,j\rangle}$ is the $(n-1)$-square matrix obtained from A by deleting the ith row and the jth column.

Proof.

First step:

$$\operatorname{per}(A_{\langle i,j\rangle}) = \operatorname{per}(A) \quad \text{whenever} \quad a_{ij} > 0. \tag{5.8}$$

To see this, let $\Delta := \{(i, j): a_{ij} > 0\}$ and consider the subspace $\mathscr{L}$ of n-square real matrices $X = (x_{ij})$ for which $x_{ij} = 0$ for all (i, j) outside of Δ. Then A is an inner point of the convex subset of $\mathscr{L}$, determined by

$$x_{ij} \geqslant 0 \quad \text{for all } i \text{ and } j, \qquad \sum_{i=1}^{n} x_{ij} = 1, \quad \text{and} \quad \sum_{j=1}^{n} x_{ij} = 1.$$

According to the Lagrange multiplier method, there exist $\vec{u}, \vec{v} \in \mathbf{R}^n$ such that the real function Φ defined on $\mathscr{L}$ by

$$\Phi(X) := \operatorname{per}(X) - \sum_{i=1}^{n} u_i\left(\sum_{j=1}^{n} x_{ij} - 1\right) - \sum_{j=1}^{n} v_j\left(\sum_{i=1}^{n} x_{ij} - 1\right) \tag{5.9}$$

attains its local minimum at A. Therefore

$$0 = \left.\frac{\partial \Phi(X)}{\partial x_{ij}}\right|_{X=A} = \operatorname{per}(A_{\langle i,j\rangle}) - u_i - v_j \qquad \text{for} \quad (i,j) \in \Delta. \tag{5.10}$$

Since

$$\sum_{i=1}^{n} \operatorname{per}(A_{\langle i,j\rangle}) a_{ij} = \sum_{j=1}^{n} \operatorname{per}(A_{\langle i,j\rangle}) a_{ij} = \operatorname{per}(A), \tag{5.11}$$

(5.10) implies

$$\operatorname{per}(A) = u_i + \sum_{j=1}^{n} a_{ij} v_j \qquad \text{for all } i, \tag{5.12}$$

$$\operatorname{per}(A) = \sum_{i=1}^{n} a_{ij} u_i + v_j \qquad \text{for all } j. \tag{5.13}$$

In vectorial notation (5.12) and (5.13) become, respectively,

$$\operatorname{per}(A)\vec{e} = \vec{u} + A\vec{v},$$

$$\operatorname{per}(A)\vec{e} = A^*\vec{u} + \vec{v}.$$

These two relations, together with $A\vec{e} = A^*\vec{e} = \vec{e}$, yield

$$A^*A\vec{v} = \vec{v} \quad \text{and} \quad AA^*\vec{u} = \vec{u}. \tag{5.14}$$

As pointed out after Lemma 5.1, both A^*A and AA^* are indecomposable doubly stochastic matrices, so that (5.14) is possible only when $\vec{u}$ and $\vec{v}$ are scalar multiples of $\vec{e}$. Then it follows from (5.10) that

$$\operatorname{per}(A_{\langle i,j\rangle}) = \text{const.} \qquad \text{for all} \quad (i,j) \in \Delta. \tag{5.15}$$

This constant must coincide with per(A), because of (5.11), proving (5.8).

Second step:

$$\operatorname{per}(A_{\langle i,j\rangle}) \geqslant \operatorname{per}(A) \qquad \text{for all } i \text{ and } j. \tag{5.16}$$

To prove this, we may assume $a_{nn} = 0$ and show per $(A_{\langle n,n\rangle}) \geqslant \operatorname{per}(A)$. Since A is fully indecomposable by Lemma 5.1, $\operatorname{per}(A_{\langle n,n\rangle}) > 0$ follows from Corollary 4.2. Then again we may assume that $a_{jj} > 0$, $j = 1,\dots,n-1$. Equation (5.8) yields

$$\operatorname{per}(A_{\langle j,j\rangle}) = \operatorname{per}(A), \qquad j = 1,\dots,n-1. \tag{5.17}$$

Let

$$A(t) := (1-t)A + tI \qquad \text{for} \quad 0 \leqslant t \leqslant 1.$$

Since any $A(t)$ is doubly stochastic, and $A = A(0)$ is a minimizing matrix,

$$0 \leqslant \frac{d}{dt}\operatorname{per}A(t)\bigg|_{t=0} = \sum_{i,j=1}^{n} (\delta_{ij} - a_{ij})\operatorname{per}(A_{\langle i,j\rangle})$$

$$= \sum_{j=1}^{n-1} \operatorname{per}(A_{\langle j,j\rangle}) + \operatorname{per}(A_{\langle n,n\rangle}) - \sum_{i,j=1}^{n} a_{ij}\operatorname{per}(A_{\langle i,j\rangle})$$

$$= \operatorname{per}(A_{\langle n,n\rangle}) - \operatorname{per}(A) \qquad \text{by (5.17) and (5.11).}$$

This proves (5.16).

Now let us turn to a proof of (5.7) on the basis of (5.8) and (5.16). It suffices to prove $\operatorname{per}(A_{\langle n,n\rangle}) \leqslant \operatorname{per}(A)$. Remark that (5.16) together with (5.11) yields that for any $1 \leqslant j \leqslant n$

$$\operatorname{per}(\vec{a}^{(1)},\dots,\vec{a}^{(j-1)},\vec{b},\vec{a}^{(j+1)},\dots,\vec{a}^{(n)}) \geqslant \operatorname{per}(A)$$
$$\text{whenever} \quad 0 \leqslant \vec{b} \text{ and } \operatorname{tr}(\vec{b}) = 1. \tag{5.18}$$

In view of the full indecomposability of A we may assume $a_{n,n-1} \equiv a_n^{(n-1)} > 0$. According to Lemma 5.2

$$\operatorname{per}(A)^2 = \operatorname{per}(\vec{a}^{(1)},\dots,\vec{a}^{(n-1)},\vec{a}^{(n)})^2$$
$$\geqslant \operatorname{per}(\vec{a}^{(1)},\dots,\vec{a}^{(n-1)},\vec{a}^{(n-1)})\operatorname{per}(\vec{a}^{(1)},\dots,\vec{a}^{(n)},\vec{a}^{(n)}). \tag{5.19}$$

Since

$$\operatorname{per}(\vec{a}^{(1)},\dots,\vec{a}^{(n)},\vec{a}^{(n)}) \geqslant \operatorname{per}(A) \qquad \text{by (5.18),}$$

it follows from (5.19) that

$$\begin{aligned}\operatorname{per}(A) &\geqslant \operatorname{per}(\vec{a}^{(1)},\dots,\vec{a}^{(n-1)},\vec{a}^{(n-1)})\\ &= \sum_{j=1}^{n-1}\operatorname{per}(A_{\langle j,n\rangle})a_{j,n-1}+\operatorname{per}(A_{\langle n,n\rangle})a_{n,n-1},\\ &\geqslant \operatorname{per}(A)(1-a_{n,n-1})+\operatorname{per}(A_{\langle n,n\rangle})a_{n,n-1},\end{aligned}$$

which proves $\operatorname{per}(A)\geqslant \operatorname{per}(A_{\langle n,n\rangle})$, because $a_{n,n-1}>0$. ■

Now we are in position to be able to settle the van der Waerden conjecture.

THEOREM 5.4. *If $A=(\vec{a}^{(1)},\dots,\vec{a}^{(n)})$ is a minimizing matrix, then $\vec{a}^{(j)}=\vec{e}/n$, $j=1,\dots,n$.*

Proof. It suffices to prove that $\vec{a}^{(n)}=\vec{e}/n$. In view of (5.11), an equivalent form of Lemma 5.3 is that for any $1\leqslant j\leqslant n$

$$\operatorname{per}(\vec{a}^{(1)},\dots,\vec{a}^{(j-1)},\vec{b},\vec{a}^{(j+1)},\dots,\vec{a}^{(n)})=\operatorname{per}(A)$$
$$\text{whenever}\quad 0\leqslant\vec{b}\ \text{and}\ \operatorname{tr}(\vec{b})=1. \tag{5.20}$$

An immediate consequence is

$$\operatorname{per}(\vec{a}^{(1)},\dots,\vec{a}^{(n-1)},\vec{a}^{(n)})^2 = \operatorname{per}(\vec{a}^{(1)},\dots,\vec{a}^{(n-1)},\vec{a}^{(n-1)})\operatorname{per}(\vec{a}^{(1)},\dots,\vec{a}^{(n-2)},\vec{a}^{(n)},\vec{a}^{(n)}).$$

Thus if all $\vec{a}^{(j)}$, $j=1,\dots,n-1$, are strictly positive, by Lemma 5.2 then $\vec{a}^{(n)}$ is a scalar multiple of $\vec{a}^{(n-1)}$; hence $\vec{a}^{(n)}=\vec{a}^{(n-1)}$. Repetition of this procedure will show that $\vec{a}^{(1)}=\cdots=\vec{a}^{(n)}$; hence

$$\vec{a}^{(n)}=n^{-1}\sum_{j=1}^{n}\vec{a}^{(j)}=\frac{\vec{e}}{n}.$$

Therefore it remains to show the existence of a minimizing matrix $B=(\vec{b}^{(1)},\dots,\vec{b}^{(n)})$ such that $\vec{b}^{(n)}=\vec{a}^{(n)}$ and all $\vec{b}^{(j)}$, $j=1,\dots,n-1$, are strictly

positive. To this end, define $C = (\vec{c}^{(1)}, \ldots, \vec{c}^{(n)})$ in the following way:

$$\vec{c}^{(1)} = \frac{\vec{a}^{(1)} + \vec{a}^{(2)}}{2} = \vec{c}^{(2)}, \qquad \vec{c}^{(j)} = \vec{a}^{(j)}, \quad j = 3, \ldots, n.$$

Obviously C is doubly stochastic, and according to (5.20)

$$\begin{aligned} \operatorname{per}(C) = {} & \tfrac{1}{4}\operatorname{per}(\vec{a}^{(1)}, \vec{a}^{(1)}, \vec{a}^{(3)}, \ldots, \vec{a}^{(n)}) \\ & + \tfrac{1}{4}\operatorname{per}(\vec{a}^{(1)}, \vec{a}^{(2)}, \vec{a}^{(3)}, \ldots, \vec{a}^{(n)}) \\ & + \tfrac{1}{4}\operatorname{per}(\vec{a}^{(2)}, \vec{a}^{(1)}, \vec{a}^{(3)}, \ldots, \vec{a}^{(n)}) \\ & + \tfrac{1}{4}\operatorname{per}(\vec{a}^{(2)}, \vec{a}^{(2)}, \vec{a}^{(3)}, \ldots, \vec{a}^{(n)}) \\ = {} & \operatorname{per}(A). \end{aligned}$$

Thus C is a minimizing matrix such that

$$I_{\vec{c}^{(1)}} = I_{\vec{c}^{(2)}} = I_{\vec{a}^{(1)}} \cup I_{\vec{a}^{(2)}},$$

$$I_{\vec{c}^{(j)}} = I_{\vec{a}^{(j)}}, \quad j = 3, \ldots, n-1, \quad \text{and} \quad \vec{c}^{(n)} = \vec{a}^{(n)}.$$

An expected minimizing matrix is reached after suitable repetition of this procedure. ∎

NOTE. A historical survey of the research on the van der Waerden conjecture is given in [53]. The final solution (Theorem 5.4) is due to Egorychev [21], who discovered the use of Lemma 5.2. We followed the exposition of Knuth [40] and van Lint [45].

III. COMPARISON OF EIGENVALUES

6. *Comparison of Eigenvalues*

A complex n-square matrix is identified with the linear map it generates on the n-dimensional Hilbert space $\mathbf{C}^n$. To each matrix $A = (a_{ij})$ is assigned the n-vector of its eigenvalues,

$$\vec{\lambda}(A) = (\lambda_1(A), \ldots, \lambda_n(A))^T, \tag{6.1}$$

arranged in any order with multiplicities counted. It is well known that there exist orthonormal vectors $\vec{x}^{(1)},\dots,\vec{x}^{(n)}$ such that

$$\langle A\vec{x}^{(j)}, \vec{x}^{(i)}\rangle = 0 \qquad \text{whenever} \quad i > j, \tag{6.2}$$

and

$$\langle A\vec{x}^{(j)}, \vec{x}^{(j)}\rangle = \lambda_j(A), \qquad j = 1,\dots,n; \tag{6.3}$$

in other words, there exists a unitary matrix W such that

$$W^*AW = (b_{ij}) \qquad \text{where} \quad b_{ij} = 0 \quad \text{for } i > j. \tag{6.4}$$

An immediate consequence is the formula

$$\operatorname{tr}(A) := \sum_{j=1}^{n} a_{jj} = \sum_{j=1}^{n} \lambda_j(A), \tag{6.5}$$

$$\det(A) := \sum_{\pi \in \mathscr{S}_n} (-1)^{\pi} \prod_{j=1}^{n} a_{j\pi_j} = \prod_{j=1}^{n} \lambda_j(A). \tag{6.6}$$

If A is Hermitian, i.e. $A = A^*$, then $\vec{\lambda}(A)$ is a real vector, and its decreasing rearrangement is denoted by

$$\vec{\lambda}^{\cdot}(A) = (\lambda^{\cdot}_1(A),\dots,\lambda^{\cdot}_n(A))^T \tag{6.7}$$

and its increasing rearrangement by

$$\vec{\lambda}_{\cdot}(A) = (\lambda_{\cdot 1}(A),\dots,\lambda_{\cdot n}(A))^T. \tag{6.8}$$

Orthonormal vectors $\vec{x}^{(1)},\dots,\vec{x}^{(n)}$ satisfying (6.2) and (6.3) are eigenvectors of A, that is,

$$A\vec{x}^{(j)} = \lambda_j(A)\vec{x}^{(j)}, \qquad j = 1,\dots,n, \tag{6.9}$$

and A is written in the form

$$A = W\operatorname{diag}\big(\vec{\lambda}(A)\big)W^*. \tag{6.10}$$

When $f(t)$ is a real-valued function defined on an interval containing all eigenvalues of A, the matrix $f(A)$ is defined by

$$f(A) = W\operatorname{diag}\Big(f\big(\vec{\lambda}(A)\big)\Big)W^*. \tag{6.11}$$

To get another expression for $f(A)$, define, for each $t \in R$, a linear map $E(A,t)$ by

$$E(A,t)\vec{x} = \sum_{\lambda_j(A) \leqslant t} \langle \vec{x}, \vec{x}^{(j)} \rangle \vec{x}^{(j)}. \tag{6.12}$$

Then $E(A,t)$ becomes the orthogonal projection to $\operatorname{span}(\vec{x}^{(j)} : \lambda_j(A) \leqslant t)$, and

$$f(A) = \int_{-\infty}^{\infty} f(t)\, dE(A,t) \tag{6.13}$$

in Stieltjes sense.

A Hermitian matrix A is said to be *positive* (or positive semidefinite), in notation $A \geqslant 0$, if $\vec{\lambda}(A) \geqslant 0$, or equivalently $\langle A\vec{x}, \vec{x} \rangle \geqslant 0$ for all $\vec{x} \in \mathbf{C}^n$. Then the order relation $B \geqslant C$ between two Hermitian matrices B, C means $B - C \geqslant 0$, or equivalently $\langle B\vec{x}, \vec{x} \rangle \geqslant \langle C\vec{x}, \vec{x} \rangle$ for all $\vec{x} \in \mathbf{C}^n$. When $A \geqslant 0$, the integral (6.13) becomes

$$f(A) = \int_{0}^{\infty} f(t)\, dE(A,t); \tag*{(6.13)$'$}$$

in particular

$$A^k = \int_{0}^{\infty} t^k\, dE(A,t), \qquad k = 1,2,\dots. \tag{6.14}$$

As a consequence, for a vector $\vec{x}$ and $r \geqslant 0$ the condition $E(A,r)\vec{x} = \vec{x}$ is equivalent to the condition

$$\langle A^k\vec{x}, \vec{x} \rangle \leqslant r^k \langle \vec{x}, \vec{x} \rangle, \qquad k = 1,2,\dots. \tag{6.15}$$

To each square matrix A are assigned several Hermitian matrices in a natural way: its *real part* $\operatorname{Re}(A) := (A + A^*)/2$, its *imaginary part* $\operatorname{Im}(A) = (A - A^*)/2\sqrt{-1}$, and its *modulus* $|A| := (A^*A)^{1/2}$. The eigenvalues of $|A|$ are called the *singular values* of A. We have

$$A = \operatorname{Re}(A) + \sqrt{-1}\operatorname{Im}(A), \tag{6.16}$$

and

$$A = U|A| \qquad \text{for a unitary matrix } U. \tag{6.17}$$

This unitary matrix U is called the *polar part* of A.

Our aim in this section is to make a comparison among $\operatorname{Re}(\vec{\lambda}(A))$, $\vec{\lambda}(\operatorname{Re}(A))$, $|\vec{\lambda}(A)|$, and $\vec{\lambda}(|A|)$ with respect to majorization, and then to establish eigenvalue analogues of (2.30)–(2.33), that is, for instance, to compare $\vec{\lambda}(A+B)$ with $\vec{\lambda}^{\cdot}(A) + \vec{\lambda}_{\cdot}(B)$ for Hermitian A, B.

LEMMA 6.1. *Let $1 \leqslant i_1 < \cdots < i_k \leqslant n$, and $\mathscr{M}_j$ and $\mathscr{N}_j$ be (complex) subspaces of $\mathbf{C}^n$ such that*

$$\mathscr{M}_j \subset \mathscr{M}_{j+1} \quad \text{with } \dim(\mathscr{M}_j) = i_j, \qquad j = 1,\dots,k, \tag{6.18}$$

and

$$\mathscr{N}_j \supset \mathscr{N}_{j+1} \quad \text{with } \dim(\mathscr{N}_j) = n - i_j + 1, \qquad j = 1,\dots,k. \tag{6.19}$$

Then there exist two orthonormal sequences $\{\vec{x}^{(1)},\dots,\vec{x}^{(k)}\}$ and $\{\vec{y}^{(1)},\dots,\vec{y}^{(k)}\}$ such that

$$\vec{x}^{(j)} \in \mathscr{M}_j \ \text{and}\ \vec{y}^{(j)} \in \mathscr{N}_j, \qquad j = 1,\dots,k, \tag{6.20}$$

and

$$\operatorname{span}(\vec{x}^{(1)},\dots,\vec{x}^{(k)}) = \operatorname{span}(\vec{y}^{(1)},\dots,\vec{y}^{(k)}). \tag{6.21}$$

Proof by induction on k. Remark first that the following estimate for the dimension results from (6.18) and (6.19):

$$\dim(\mathscr{M}_j \cap \mathscr{N}_1) \geqslant j, \qquad j = 1,\dots,k. \tag{6.22}$$

In fact,

$$\begin{aligned}\dim(\mathscr{M}_j \cap \mathscr{N}_1) &= \dim(\mathscr{N}_1) + \dim(\mathscr{M}_j) - \dim(\mathscr{M}_j + \mathscr{N}_1)\\ &\geqslant n - i_1 + 1 + i_j - n\\ &\geqslant n + 1 + j - 1 - n = j.\end{aligned}$$

If $k = 1$, (6.22) with $k = 1$ guarantees the existence of a normalized vector $\vec{x}_1 \equiv \vec{y}_1 \in \mathscr{M}_1 \cap \mathscr{N}_1$. Suppose that the assertion is true for $k-1$. Then applied to $\mathscr{M}_j$ and $\mathscr{N}_j$, $j = 2,\dots,k$, the assumption guarantees the existence of two orthonormal sequences $\{\vec{u}^{(2)},\dots,\vec{u}^{(k)}\}$ and $\{\vec{v}^{(2)},\dots,\vec{v}^{(k)}\}$ such that $\vec{u}^{(j)} \in \mathscr{M}_j$, $\vec{v}^{(j)} \in \mathscr{N}_j$, $j = 2,\dots,k$, and

$$\operatorname{span}(\vec{u}^{(2)},\dots,\vec{u}^{(k)}) = \operatorname{span}(\vec{v}^{(2)},\dots,\vec{v}^{(k)}) \equiv \mathscr{L}. \tag{6.23}$$

Take any nonzero $\vec{w}^{(0)} \in \mathscr{M}_1 \cap \mathscr{N}_1$ whose existence follows from (6.22) as above. If $\vec{w}^{(0)} \in \mathscr{L}$, denote by k_1 the minimum of j such that $\vec{w}^{(0)} \in \operatorname{span}(\vec{u}^{(2)},\dots,\vec{u}^{(j)})$. Then $1 < k_1 \leqslant k$ and

$$\operatorname{span}(\vec{u}^{(2)},\dots,\vec{u}^{(k_1)}) = \operatorname{span}(\vec{w}^{(0)},\vec{u}^{(2)},\dots,\vec{u}^{(k_1-1)}). \tag{6.24}$$

Orthonormalize $\vec{w}^{(0)}, \vec{u}^{(2)},\dots,\vec{u}^{(k_1-1)}$ from left to right to get $\vec{x}^{(1)}, \vec{x}^{(2)},\dots,\vec{x}^{(k_1-1)}$. Since $\vec{w}^{(0)} \in \mathscr{M}_1$ and $\vec{u}^{(j)} \in \mathscr{M}_j$, $j = 2,\dots,k-1$, (6.18) implies $\vec{x}^{(j)} \in \mathscr{M}_j$, $j = 1,\dots,k_1-1$, and then by (6.24)

$$\operatorname{span}(\vec{u}^{(2)},\dots,\vec{u}^{(k_1)}) = \operatorname{span}(\vec{x}^{(1)},\dots,\vec{x}^{(k_1-1)}). \tag{6.25}$$

Since $\dim(\mathscr{M}_{k_1} \cap \mathscr{N}_1) \geqslant k_1$ by (6.22), there exists nonzero $\vec{w}^{(1)} \in \mathscr{M}_{k_1} \cap \mathscr{N}_1$ such that

$$\langle \vec{x}^{(j)}, \vec{w}^{(1)} \rangle = 0, \qquad j = 1,\dots,k_1-1, \tag{6.26}$$

or equivalently, by (6.25),

$$\langle \vec{u}^{(j)}, \vec{w}^{(1)} \rangle = 0, \qquad j = 2,\dots,k_1. \tag{6.27}$$

If $\vec{w}^{(1)} \in \mathscr{L}$, by (6.23) and (6.27) $\vec{w}^{(1)}$ is in $\operatorname{span}(\vec{u}^{(k_1+1)},\dots,\vec{u}^{(k)})$. Denote by k_2 the minimum of j such that $\vec{w}^{(1)} \in \operatorname{span}(\vec{u}^{(k_1+1)},\dots,\vec{u}^{(j)})$, and orthonormalize $\vec{w}^{(1)}, \vec{u}^{(k_1+1)},\dots,\vec{u}^{(k_2-1)}$ from left to right to get $\vec{x}^{(k_1)},\dots,\vec{x}^{(k_2-1)}$ such

that

$$\operatorname{span}(\vec{w}^{(1)}, \vec{u}^{(k_1+1)}, \dots, \vec{u}^{(k_2-1)}) = \operatorname{span}(\vec{x}^{(k_1)}, \dots, \vec{x}^{(k_2-1)}).$$

After a finite number l of repetitions of this procedure, we arrive at the situations that $\{\vec{x}^{(1)}, \dots, \vec{x}^{(l-1)}\}$ is an orthonormal sequence such that $\vec{x}^{(j)} \in \mathscr{M}_j$ and

$$\operatorname{span}(\vec{x}^{(1)}, \dots, \vec{x}^{(l-1)}) = \operatorname{span}(\vec{u}^{(2)}, \dots, \vec{u}^{(l)})$$

and that there exists nonzero $\vec{w} \in \mathscr{M}_l \cap \mathscr{N}_1$ that does not belong to $\mathscr{L}$. Orthonormalize $\vec{w}, \vec{u}^{(l+1)}, \dots, \vec{u}^{(k)}$ from left to right to get $\vec{x}^{(l)}, \dots, \vec{x}^{(k)}$. Clearly $\vec{x}^{(j)} \in \mathscr{M}_j$, $j = 1, \dots, k$, and

$$\operatorname{span}(\vec{x}^{(1)}, \dots, \vec{x}^{(k)}) = \operatorname{span}(\vec{w}, \mathscr{L}).$$

Finally orthonormalize $\vec{w}, \vec{v}^{(2)}, \dots, \vec{v}^{(k)}$ from right to left to get $\vec{y}^{(1)}, \dots, \vec{y}^{(k)}$. Since $\vec{w} \in \mathscr{N}_1$ and $\vec{v}^{(j)} \in \mathscr{N}_j$, (6.19) implies that $\vec{y}^{(j)} \in \mathscr{N}_j$, $j = 1, \dots, k$, and

$$\operatorname{span}(\vec{y}^{(1)}, \dots, \vec{y}^{(k)}) = \operatorname{span}(\vec{w}, \mathscr{L}).$$

This completes the induction. ∎

Given a linear map A and a subspace $\mathscr{L}$, let us denote by $A_{\mathscr{L}}$ the *compression* of A to $\mathscr{L}$, that is, $A_{\mathscr{L}}$ is a linear map on $\mathscr{L}$ defined by

$$A_{\mathscr{L}}\vec{x} = PA\vec{x} \qquad \text{for} \quad \vec{x} \in \mathscr{L},$$

where P is the orthogonal projection to $\mathscr{L}$.

LEMMA 6.2. *Let A be a Hermitian matrix and $\vec{a}^{(j)}$, $j = 1, \dots, n$, its orthonormal eigenvectors such that*

$$A\vec{a}^{(j)} = \lambda_j^{\downarrow}(A)\vec{a}^{(j)}, \qquad j = 1, \dots, n.$$

Let

$$\hat{\mathscr{M}}_j := \operatorname{span}(\vec{a}^{(1)}, \dots, \vec{a}^{(j)}), \qquad j = 1, \dots, n.$$

For any choice of k indices such that $1 \leqslant i_1 < \cdots < i_k \leqslant n$ *and any orthonormal sequence* $\vec{x}^{(j)} \in \hat{\mathscr{M}}_{i_j}$, $j = 1, \ldots, k$, *the following inequalities hold*:

$$\lambda^{\cdot}_j(A_{\mathscr{L}}) \geqslant \lambda^{\cdot}_{i_j}(A), \qquad j = 1, \ldots, k, \tag{6.28}$$

where

$$\mathscr{L} = \operatorname{span}(\vec{x}^{(1)}, \ldots, \vec{x}^{(k)}).$$

Proof. Let $\vec{b}^{(1)}, \ldots, \vec{b}^{(k)}$ be orthonormal eigenvectors of $A_{\mathscr{L}}$ such that

$$A_{\mathscr{L}} \vec{b}^{(j)} = \lambda^{\cdot}_j(A_{\mathscr{L}}) \vec{b}^{(j)}, \qquad j = 1, \ldots, k.$$

As in (6.22), it is possible to take a normalized vector $\vec{x}$ in $\operatorname{span}(\vec{x}^{(1)}, \ldots, \vec{x}^{(j)}) \cap \operatorname{span}(\vec{b}^{(j)}, \ldots, \vec{b}^{(k)})$. Then

$$\begin{aligned} \lambda^{\cdot}_j(A_{\mathscr{L}}) &= \max_{\substack{\vec{y} \in \operatorname{span}(\vec{b}^{(j)}, \ldots, \vec{b}^{(k)}) \\ \|\vec{y}\| = 1}} \langle A\vec{y}, \vec{y} \rangle \\ &\geqslant \langle A\vec{x}, \vec{x} \rangle \\ &\geqslant \min_{\substack{\vec{y} \in \hat{\mathscr{M}}_{i_j} \\ \|\vec{y}\| = 1}} \langle A\vec{y}, \vec{y} \rangle = \lambda^{\cdot}_{i_j}(A). \end{aligned}$$ ■

THEOREM 6.3. *Suppose that a function* $\Phi(t_1, \ldots, t_k)$ *defined for* $\alpha \leqslant t_j \leqslant \beta$, $j = 1, \ldots, k$, *is permutation-invariant and monotone increasing. Then for any choice of k indices such that* $1 \leqslant i_1 < \cdots < i_k \leqslant n$ *and any (n-square) Hermitian matrix* A, *all of whose eigenvalues are contained in the interval* $[\alpha, \beta]$, *the following identities hold*:

$$\Phi\left(\lambda^{\cdot}_{i_1}(A), \ldots, \lambda^{\cdot}_{i_k}(A)\right) = \max_{\substack{\mathscr{M}_1 \subset \cdots \subset \mathscr{M}_k \\ \dim(\mathscr{M}_j) = i_j}} \min_{\substack{\mathscr{M}_j \ni \vec{x}^{(j)} \text{ O.N.} \\ \mathscr{L} = \operatorname{span}(\vec{x}^{(1)}, \ldots, \vec{x}^{(k)})}} \Phi\left(\vec{\lambda}(A_{\mathscr{L}})\right). \tag{6.29}$$

Proof. That the right side of (6.29) is not smaller than the left side follows from Lemma 6.2 via the monotony of Φ.

To see the reversed inequality, using the notation of Lemma 6.2, let $\mathscr{N}_j$ be the orthocomplement of $\hat{\mathscr{M}}_{i_j - 1}$. Then $\mathscr{N}_j \supset \mathscr{N}_{j+1}$ with $\dim(\mathscr{N}_j) = n - i_j + 1$, and $\mathscr{N}_j$ plays the same role for $-A$ as $\hat{\mathscr{M}}_{n - i_j + 1}$ does for A.

Now take any sequence of subspaces $\mathscr{M}_1 \subset \mathscr{M}_2 \subset \dots \subset \mathscr{M}_k$ with $\dim(\mathscr{M}_j) = i_j$, $j = 1,\dots,k$. According to Lemma 6.1 there exist orthonormal $\{\vec{x}^{(1)},\dots,\vec{x}^{(k)}\}$ and $\{\vec{y}^{(1)},\dots,\vec{y}^{(k)}\}$ satisfying (6.20) and (6.21). Let

$$\mathscr{L} := \operatorname{span}(\vec{x}^{(1)},\dots,\vec{x}^{(k)}) = \operatorname{span}(\vec{y}^{(1)},\dots,\vec{y}^{(k)}).$$

Apply Lemma 6.2 to $-A$ to get

$$\begin{aligned} -\lambda^{\downarrow}_j(A_{\mathscr{L}}) &= \lambda^{\downarrow}_{k-j+1}(-A_{\mathscr{L}}) \\ &\geqslant \lambda^{\downarrow}_{n-i_j+1}(-A) = -\lambda^{\downarrow}_{i_j}(A). \end{aligned}$$

Then the monotony of Φ yields

$$\Phi\big(\vec{\lambda}(A_{\mathscr{L}})\big) \leqslant \Phi\big(\lambda^{\downarrow}_{i_1}(A),\dots,\lambda^{\downarrow}_{i_k}(A)\big).$$

This observation shows that the right side of (6.29) is not greater than the left side. ∎

Corollary 6.4. *If A is Hermitian,*

$$\begin{aligned} \lambda^{\downarrow}_j(A) &= \max_{\dim(\mathscr{M})=j} \min_{\substack{\vec{x}\in\mathscr{M} \\ \|\vec{x}\|=1}} \langle A\vec{x},\vec{x}\rangle \\ &= \min_{\dim(\mathscr{N})=n-j+1} \max_{\substack{\vec{y}\in\mathscr{N} \\ \|\vec{y}\|=1}} \langle A\vec{y},\vec{y}\rangle, \qquad j=1,\dots,n. \end{aligned} \tag{6.30}$$

Corollary 6.5. *Let A, B be Hermitian.*

(a) *If $A \geqslant B$, then*

$$\lambda^{\downarrow}_j(A) \geqslant \lambda^{\downarrow}_j(B), \qquad j=1,\dots,n. \tag{6.31}$$

(b) *If a subspace $\mathscr{L}$ is of dimension k, then*

$$\lambda^{\downarrow}_j(A) \geqslant \lambda^{\downarrow}_j(A_{\mathscr{L}}), \qquad j=1,\dots,k. \tag{6.32}$$

(c) *If $A \geqslant 0$ and P is an orthogonal projection,*

$$\lambda^{\cdot}_j(A) \geqslant \lambda^{\cdot}_j(PAP), \qquad j = 1, \ldots, n. \tag{6.33}$$

Proof. (a) is immediate from (6.29).

(b): $\lambda^{\cdot}_j(A_{\mathscr{L}})$ is obtained by restricting $\mathscr{M}$ to be a subspace of $\mathscr{L}$ in (6.30).

(c):
$$\begin{aligned}\lambda^{\cdot}_j(PAP) &= \lambda^{\cdot}_j(PA^{1/2}A^{1/2}P)\\ &= \lambda^{\cdot}_j(A^{1/2}PA^{1/2})\\ &\leqslant \lambda^{\cdot}_j(A) \qquad \text{by (6.31)}. \qquad \blacksquare\end{aligned}$$

Let us insert here a promised proof to a result used for Lemma 5.2.

COROLLARY 6.6. *Suppose that an n-square Hermitian matrix A ($n \geqslant 2$) and an n-vector $\vec{a}$ satisfy $\langle A\vec{a}, \vec{a}\rangle > 0$. Then the following conditions are mutually equivalent:*

(i) $\lambda^{\cdot}_2(A) < 0$.
(ii) $|\langle A\vec{a}, \vec{x}\rangle|^2 > \langle A\vec{a}, \vec{a}\rangle\langle A\vec{x}, \vec{x}\rangle$ *for all $\vec{x}$ linearly independent of $\vec{a}$.*
(iii) $\langle A\vec{x}, \vec{x}\rangle < 0$ *whenever* $\langle A\vec{a}, \vec{x}\rangle = 0$ *and* $\vec{x} \neq 0$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $\vec{x}$ is not a scalar multiple of $\vec{a}$, but

$$|\langle A\vec{a}, \vec{x}\rangle|^2 \leqslant \langle A\vec{a}, \vec{a}\rangle\langle A\vec{x}, \vec{x}\rangle.$$

Then for any α, $\beta \in \mathbf{C}$

$$\langle A(\alpha\vec{a} + \beta\vec{x}), \alpha\vec{a} + \beta\vec{x}\rangle \geqslant 0,$$

which implies $A_{\mathscr{L}} \geqslant 0$, where $\mathscr{L} = \operatorname{span}(\vec{a}, \vec{x})$. It follows from (6.32) that

$$\lambda^{\cdot}_2(A) \geqslant \lambda^{\cdot}_2(A_{\mathscr{L}}) \geqslant 0,$$

contradicting (i).

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Suppose that $\lambda^{\cdot}_2(A) \geqslant 0$. Then there exists an orthogonal projection P of rank 2 such that $PAP \geqslant 0$. Take a nonzero vector $\vec{x}$ such that $P\vec{x} = \vec{x}$ and $\langle PA\vec{a}, \vec{x}\rangle = 0$, which is possible because the range of P is two-dimensional. Then $\langle A\vec{a}, \vec{x}\rangle = 0$, while $\langle A\vec{x}, \vec{x}\rangle \geqslant 0$, contradicting (iii). ■

Let us begin with a comparison of $\vec{\lambda}(|A|)$, $\vec{\lambda}(\mathrm{Re}(A))$, and $\mathrm{Re}(\vec{\lambda}(A))$. A comparison of $\vec{\lambda}(|A|)$ and $|\vec{\lambda}(A)|$ will be given later.

THEOREM 6.7. *Let A be an n-square matrix. Then*

$$\mathrm{Re}\big(\vec{\lambda}(A)\big) \prec \vec{\lambda}^{\cdot}(\mathrm{Re}(A)) \leqslant \vec{\lambda}^{\cdot}(|A|). \tag{6.34}$$

Proof. To see the majorization in (6.34), according to (6.3), take an orthonormal sequence $\vec{x}^{(1)},\dots,\vec{x}^{(n)}$ such that

$$\langle A\vec{x}^{(j)}, \vec{x}^{(j)}\rangle = \lambda_j(A), \qquad j = 1,\dots,n.$$

We may assume

$$\mathrm{Re}(\lambda_1(A)) \geqslant \dots \geqslant \mathrm{Re}(\lambda_n(A)).$$

Then for any k, with $\mathscr{L} = \mathrm{span}(\vec{x}^{(1)},\dots,\vec{x}^{(k)})$,

$$\begin{aligned}\sum_{j=1}^{k} \mathrm{Re}(\lambda_j(A)) &= \sum_{j=1}^{k} \langle \mathrm{Re}(A)\vec{x}^{(j)}, \vec{x}^{(j)}\rangle \\ &= \sum_{j=1}^{k} \lambda_j((\mathrm{Re}(A))_{\mathscr{L}}) \\ &\leqslant \sum_{j=1}^{k} \lambda^{\cdot}_j(\mathrm{Re}(A)) \qquad \text{by (6.32)},\end{aligned}$$

and equality occurs for the case $k = n$. Therefore $\mathrm{Re}(\vec{\lambda}(A)) \prec \vec{\lambda}(\mathrm{Re}(A))$.

Turning to a proof of the inequality in (6.34), let $\vec{b}^{(1)},\dots,\vec{b}^{(n)}$ be orthonormalized eigenvectors of $|A|$ such that

$$|A|\vec{b}^{(j)} = \lambda^{\cdot}_j(|A|)\vec{b}^{(j)}, \qquad j = 1,\dots,n,$$

and let

$$\mathscr{N}_j = \operatorname{span}(\vec{b}^{(j)}, \dots, \vec{b}^{(n)}).$$

Then in view of (6.30)

$$\begin{aligned}\lambda^{\boldsymbol{\cdot}}_j(\operatorname{Re}(A)) &\leqslant \max_{\substack{\vec{y} \in \mathscr{N}_j \\ \|\vec{y}\| = 1}} \langle \operatorname{Re}(A)\vec{y}, \vec{y} \rangle \\ &\leqslant \max_{\substack{\vec{y} \in \mathscr{N}_j \\ \|\vec{y}\| = 1}} \|A\vec{y}\| \leqslant \lambda^{\boldsymbol{\cdot}}_j(|A|). \qquad \blacksquare\end{aligned}$$

An equivalent form of the inequality in (6.34) is the following matrix inequality:

$$\operatorname{Re}(A) \leqslant W^*|A|W \qquad \text{for some unitary } W. \tag{6.35}$$

This inequality can be used to establish another important inequality for a pair A, B:

$$|A + B| \leqslant W_1^*|A|W_1 + W_2^*|B|W_2 \qquad \text{for some unitary } W_1, W_2. \tag{6.36}$$

In fact, let U be the polar part of $A + B$. Then

$$|A + B| = \operatorname{Re}(U^*A) + \operatorname{Re}(U^*B).$$

According to (6.35) there exist unitary matrices W_1, W_2 such that

$$\operatorname{Re}(U^*A) \leqslant W_1^*|U^*A|W_1 = W_1^*|A|W_1$$

and similarly

$$\operatorname{Re}(U^*B) \leqslant W_2^*|B|W_2.$$

Now let us turn to the comparison of $\vec{\lambda}(A + B)$, $\vec{\lambda}(A)$, and $\vec{\lambda}(B)$ for Hermitian A, B.

THEOREM 6.8. *If A and B are Hermitian, then*

$$\vec{\lambda}^{\boldsymbol{\cdot}}(A + B) - \vec{\lambda}^{\boldsymbol{\cdot}}(B) \prec \vec{\lambda}^{\boldsymbol{\cdot}}(A). \tag{6.37}$$

Proof. It suffices to prove that for any choice of k indices such that $1 \leqslant i_1 < \cdots < i_k \leqslant n$

$$\sum_{j=1}^{k} \lambda^{\cdot}_{i_j}(A+B) \leqslant \sum_{j=1}^{k} \lambda^{\cdot}_{j}(A) + \sum_{j=1}^{k} \lambda^{\cdot}_{i_j}(B). \tag{6.38}$$

Let

$$\Phi(t_1,\dots,t_k) = \sum_{j=1}^{k} t_j.$$

For any k-dimensional subspace $\mathscr{L}$

$$\begin{aligned}\Phi\big(\vec{\lambda}((A+B)_{\mathscr{L}})\big) &= \Phi\big(\vec{\lambda}(A_{\mathscr{L}})\big) + \Phi\big(\vec{\lambda}(B_{\mathscr{L}})\big)\\ &\leqslant \sum_{j=1}^{k} \lambda^{\cdot}_{j}(A) + \Phi\big(\vec{\lambda}(B_{\mathscr{L}})\big) \qquad \text{by (6.32).}\end{aligned}$$

Now appeal to Theorem 6.3 to get (6.38). ∎

(6.37) yields, via suitable substitutions, an eigenvalue analogue of (2.30):

$$\vec{\lambda}^{\cdot}(A) + \vec{\lambda}_{\cdot}(B) \prec \vec{\lambda}(A+B) \prec \vec{\lambda}^{\cdot}(A) + \vec{\lambda}^{\cdot}(B) \qquad \text{for Hermitian } A, B. \tag{6.39}$$

THEOREM 6.9. *Suppose that $f(t)$ is a nondecreasing concave function on $[0,\infty)$ with $f(0)=0$. Then for any square matrices A, B the following holds*:

$$\vec{\lambda}^{\cdot}(f(|A+B|)) - \vec{\lambda}^{\cdot}(f(|B|)) \prec \vec{\lambda}^{\cdot}(f(|A|)). \tag{6.40}$$

Proof. Fix k, consider a function

$$\Phi(t_1,\dots,t_k) := \sum_{j=1}^{k} f(t_j) \qquad \text{for} \quad t_j \geqslant 0, \quad j=1,\dots,k,$$

and take any k-dimensional subspace $\mathscr{L}$. Then with the notation of (6.36), by (6.31)

$$\Phi\big(\vec{\lambda}(|A+B|_{\mathscr{L}})\big) \leqslant \Phi\big(\vec{\lambda}\left[(W_1^*|A|W_1)_{\mathscr{L}} + (W_2^*|B|W_2)_{\mathscr{L}}\right]\big).$$

Then in view of the concavity of f it follows from Theorem 7.5 proved in the next section that

$$\Phi\left(\vec{\lambda}(|A+B|_{\mathscr{L}})\right) \leqslant \Phi\left(\vec{\lambda}(|A|_{W_1\mathscr{L}})\right) + \Phi\left(\vec{\lambda}(|B|_{W_2\mathscr{L}})\right)$$

$$\leqslant \Phi(\lambda_1^{\cdot}(|A|),\ldots,\lambda_k^{\cdot}(|A|)) + \Phi\left(\vec{\lambda}(|B|_{W_2\mathscr{L}})\right) \qquad \text{by (6.32)}.$$

Now appeal to Theorem 6.3, with the presence of W_2 taken into account, to establish that for any choice of k indices such that $1 \leqslant i_1 < \cdots < i_k \leqslant n$

$$\sum_{j=1}^{k} f\left(\lambda_{i_j}^{\cdot}(|A+B|)\right) \leqslant \sum_{j=1}^{k} f\left(\lambda_j^{\cdot}(|A|)\right) + \sum_{j=1}^{k} f\left(\lambda_{i_j}^{\cdot}(|B|)\right). \qquad \blacksquare$$

To obtain an eigenvalue analogue of (2.31), we need an inequality of Schwarz type for determinants.

LEMMA 6.10. *Let* $1 \leqslant k \leqslant n$. *For any* $2k$ *n-vectors* $\vec{u}^{(1)},\ldots,\vec{u}^{(k)}$, $\vec{v}^{(1)},\ldots,\vec{v}^{(k)}$ *the following inequality holds for the determinants of k-square matrices*:

$$\left|\det\left[\langle \vec{u}^{(i)}, \vec{v}^{(j)}\rangle\right]\right|^2 \leqslant \det\left[\langle \vec{u}^{(i)}, \vec{u}^{(j)}\rangle\right] \det\left[\langle \vec{v}^{(i)}, \vec{v}^{(j)}\rangle\right]. \tag{6.41}$$

Proof. Denote by S and T linear maps from $\mathbf{C}^k$ to $\mathbf{C}^n$ defined by

$$S\vec{e}_{(j)} = \vec{u}^{(j)} \text{ and } T\vec{e}_{(j)} = \vec{v}^{(j)}, \qquad j = 1,\ldots,k,$$

where

$$\vec{e}_{(j)} := (0,\ldots,0,\overset{(j)}{1},0,\ldots,0)^T \in \mathbf{C}^k.$$

Then the inequality (6.41) has the following equivalent form:

$$|\det(T^*S)|^2 \leqslant \det(S^*S)\det(T^*T). \tag{6.41$'$}$$

Since the orthogonal projection P to the range of S is at most of rank k, there is a partial isometric linear map V from $\mathbf{C}^k$ to $\mathbf{C}^n$ such that $P = VV^*$. Then

$$\begin{aligned}
|\det(T^*S)|^2 &= |\det(T^*VV^*S)|^2 \\
&= |\det(T^*V)\det(V^*S)|^2 \\
&= \det(T^*V)\det(V^*T)\det(S^*V)\det(V^*S) \\
&= \det(T^*VV^*T)\det(S^*VV^*S) \\
&= \det(T^*PT)\det(S^*S) \\
&\leqslant \det(T^*T)\det(S^*S),
\end{aligned}$$

because $T^*PT \leqslant T^*T$ implies, by (6.31),

$$\begin{aligned}
\det(T^*PT) &= \prod_{j=1}^{k} \lambda_j^{\cdot}(T^*PT) \\
&\leqslant \prod_{j=1}^{k} \lambda_j^{\cdot}(T^*T) = \det(T^*T). \qquad \blacksquare
\end{aligned}$$

Given $0 \leqslant \vec{x}, \vec{y}, \vec{z} \in \mathbf{R}^n$, let us write $\log \vec{x} \underset{\cdot}{\prec} \log \vec{y}$ to mean

$$\prod_{j=1}^{k} x_j^{\cdot} \leqslant \prod_{j=1}^{k} y_j^{\cdot}, \qquad k = 1, \dots, n. \tag{6.42}$$

If, in addition to (6.42), equality occurs at $k = n$, we write $\log \vec{x} \prec \log \vec{y}$. In a similar way, let us write

$$\log \vec{x} - \log \vec{z} \underset{\cdot}{\prec} \log \vec{y}$$

to mean that

$$\prod_{j=1}^{k} x_{i_j} \leqslant \prod_{j=1}^{k} y_j^{\cdot} \prod_{j=1}^{k} z_{i_j} \qquad \text{for all} \quad i_1 < \cdots < i_k. \tag{6.43}$$

If, in addition to (6.43), equality occurs at $k = n$, we write

$$\log \vec{x} - \log \vec{z} \prec \log \vec{y}.$$

The notation

$$\log \vec{x} \underset{\cdot}{\prec} \log \vec{y}^{\,\cdot} + \log \vec{z}^{\,\cdot}$$

should be understood in the corresponding way. Remark that if $\vec{x}$, $\vec{y}$ and $\vec{z}$ are strictly positive, this conventional notation coincides with the usual one among well-defined vectors $\log \vec{x}$, $\log \vec{y}$ and $\log \vec{z}$.

THEOREM 6.11. *For the eigenvalues of an n-square matrix A and those of $|A|$ the following majorization holds*:

$$\log\left|\vec{\lambda}(A)\right| \prec \log \vec{\lambda}(|A|). \tag{6.44}$$

Proof. Take, according to (6.3), an orthonormal sequence $\vec{x}^{(1)},\dots,\vec{x}^{(n)}$ such that

$$\langle A\vec{x}^{(j)}, \vec{x}^{(i)}\rangle = 0 \qquad \text{for} \quad i > j,$$

$$\langle A\vec{x}^{(j)}, \vec{x}^{(j)}\rangle = \lambda_j(A), \qquad j = 1,\dots,n.$$

We may assume

$$|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)|.$$

Then for any k, with $\mathscr{L} = \operatorname{span}(\vec{x}^{(1)},\dots,\vec{x}^{(k)})$,

$$\begin{aligned}
\left\{\prod_{j=1}^{k} |\lambda_j(A)|\right\}^2 &= \left|\det\left[\langle A\vec{x}^{(i)}, \vec{x}^{(j)}\rangle\right]_{1\leqslant i,j\leqslant k}\right|^2 \\
&\leqslant \det\left[\langle A^*A\vec{x}^{(i)}, \vec{x}^{(j)}\rangle\right]_{1\leqslant i,j\leqslant k} && \text{by (6.41)} \\
&= \prod_{j=1}^{k} \lambda_j((A^*A)_{\mathscr{L}}) \leqslant \prod_{j=1}^{k} \lambda_j^{\cdot}(A^*A) && \text{by (6.32)} \\
&= \left\{\prod_{j=1}^{k} \lambda_j^{\cdot}(|A|)\right\}^2,
\end{aligned}$$

and equality occurs at $k = n$. ■

COROLLARY 6.12. *Let $f(t)$ be a real-valued function defined on $[0, \infty)$. If $f(e^t)$ is convex, then*

$$f\big(|\vec{\lambda}(A)|\big) \underset{\cdot}{\prec} f\big(\vec{\lambda}(|A|)\big) \qquad \textit{for all n-square } A. \tag{6.45}$$

This follows immediately from Theorem 6.11 and Corollary 2.2.

THEOREM 6.13. *For any n-square matrices A, B the following majorization holds:*

$$\log \vec{\lambda}^{\cdot}(|AB|) - \log \vec{\lambda}^{\cdot}(|B|) \prec \log \vec{\lambda}(|A|). \tag{6.46}$$

Proof. Fix k, and let

$$\Phi(t_1, \dots, t_k) = \prod_{j=1}^{k} t_j \qquad \text{for} \quad t_j \geqslant 0, \quad j = 1, \dots, k.$$

Take any orthonormal $\vec{x}^{(1)}, \dots, \vec{x}^{(k)}$ and let $\mathscr{L} = \operatorname{span}(\vec{x}^{(1)}, \dots, \vec{x}^{(k)})$. Then, with the polar part U of AB,

$$\begin{aligned}
\Phi\big(\vec{\lambda}(|AB|_{\mathscr{L}})\big)^2 &= \big|\det[\langle AB\vec{x}^{(i)}, U\vec{x}^{(j)}\rangle]\,\big|^2 \\
&\leqslant \det[\langle AA^*U\vec{x}^{(i)}, U\vec{x}^{(j)}\rangle]\det[\langle B^*B\vec{x}^{(i)}, \vec{x}^{(j)}\rangle] \qquad \text{by (6.41)} \\
&= \det[(AA^*)_{U\mathscr{L}}]\det\big(|B|^2_{\mathscr{L}}\big) \\
&\leqslant \left\{\prod_{j=1}^{k} \lambda^{\cdot}_j(|A|)\right\}^2 \det\big(|B|^2_{\mathscr{L}}\big) \qquad \text{by (6.32).}
\end{aligned}$$

Now appeal to Theorem 6.3 to get

$$\prod_{j=1}^{k} \lambda^{\cdot}_{i_j}(|AB|) \leqslant \prod_{j=1}^{k} \lambda^{\cdot}_j(|A|) \cdot \prod_{j=1}^{k} \lambda^{\cdot}_{i_j}(|B|).$$

For $k = n$ equality occurs. ∎

(6.46) yields, via suitable substitutions,

$$\log \vec{\lambda}^{\cdot}(|A|)+\log \vec{\lambda}_{\cdot}(|B|) \prec \log \vec{\lambda}(|AB|) \prec \log \vec{\lambda}^{\cdot}(|A|)+\log \vec{\lambda}^{\cdot}(|B|). \quad (6.47)$$

COROLLARY 6.14. *All eigenvalues of the product of any two positive matrices A, B are nonnegative, and*

$$\log \vec{\lambda}^{\cdot}(A)+\log \vec{\lambda}_{\cdot}(B) \prec \log \vec{\lambda}(AB) \prec \log \vec{\lambda}^{\cdot}(A)+\log \vec{\lambda}^{\cdot}(B). \quad (6.48)$$

Proof. We may assume invertibility of B. Since

$$AB = B^{-1/2}(B^{1/2}AB^{1/2})B^{1/2},$$

AB and the positive matrix $B^{1/2}AB^{1/2}$ are similar and have the same set of eigenvalues. Since

$$B^{1/2}AB^{1/2} = |A^{1/2}B^{1/2}|^2,$$

apply (6.47) with $A^{1/2}$ and $B^{1/2}$ instead of A and B respectively, and multiply both sides by 2 to get (6.48). ∎

(6.48) is an eigenvalue analogue of (2.31).

COROLLARY 6.15. *Let $f(t)$ be a real-valued function defined on $[0,\infty)$. If $f(e^t)$ is convex, then*

$$f\left(\vec{\lambda}^{\cdot}(|A|)\vec{\lambda}_{\cdot}(|B|)\right) \underset{\cdot}{\prec} f\left(\vec{\lambda}(|AB|)\right) \underset{\cdot}{\prec} f\left(\vec{\lambda}^{\cdot}(|A|)\vec{\lambda}^{\cdot}(|B|)\right). \quad (6.49)$$

In case of positive A, B, the modulus $|AB|$ can be replaced by AB.

With $f(t)=t$, an eigenvalue analogue of (2.31) follows:

$$\vec{\lambda}^{\cdot}(A)\vec{\lambda}_{\cdot}(B) \underset{\cdot}{\prec} \vec{\lambda}(AB) \underset{\cdot}{\prec} \vec{\lambda}^{\cdot}(A)\vec{\lambda}^{\cdot}(B) \qquad \text{for} \quad A, B \geqslant 0, \quad (6.50)$$

which implies, just as in vector case (2.34),

$$\left\langle \vec{\lambda}^{\cdot}(A), \vec{\lambda}_{\cdot}(B)\right\rangle \leqslant \operatorname{tr}(AB) \leqslant \left\langle \vec{\lambda}^{\cdot}(A), \vec{\lambda}^{\cdot}(B)\right\rangle \qquad \text{for Hermitian } A, B. \quad (6.51)$$

Our final goal in this section is to establish an eigenvalue analogue of (2.32) and (2.33). But as the space of Hermitian matrices does not become a lattice with respect to the usual order $\leqslant$, it is necessary to introduce reasonable definitions of the symbols $A \vee B$ and $A \wedge B$ for Hermitian matrices A, B.

Given two Hermitian A, B, let us write

$$A \overset{\sigma}{\leqslant} B$$

(*spectral order*) if $f(A) \leqslant f(B)$ for all nondecreasing functions $f(t)$ on $\mathbf{R}$. Obviously

$$A \overset{\sigma}{\leqslant} B \quad \text{implies} \quad A \leqslant B,$$

but not conversely. The following are immediate from definitions;

$$A \overset{\sigma}{\leqslant} B \quad \text{if and only if} \quad A + \alpha I \overset{\sigma}{\leqslant} B + \alpha I \text{ for } \alpha \in \mathbf{R}. \tag{6.52}$$

$$A \overset{\sigma}{\leqslant} B \quad \text{if and only if} \quad -A \overset{\sigma}{\geqslant} -B. \tag{6.53}$$

LEMMA 6.15. *The space of Hermitian matrices becomes a lattice with respect to spectral order. More explicitly, if A, B are invertible positive matrices, the supremum $A \vee B$ is obtained as the increasing limit of $\{(A^k + B^k)/2\}^{1/k}$ as $k \to \infty$, while the infimum $A \wedge B$ is the decreasing limit of $\{(A^{-k} + B^{-k})/2\}^{-1/k}$.*

Proof. Let us use the fact, proved in the next section, that for any $0 < p \leqslant 1$, the map $0 \leqslant X \mapsto X^p$ is monotone increasing and concave with respect to the usual order. Now take $k > l \geqslant 1$. The concavity yields

$$\left\{\frac{A^k + B^k}{2}\right\}^{l/k} \geqslant \frac{A^l + B^l}{2}$$

and further

$$\left\{\frac{A^k + B^k}{2}\right\}^{1/k} \geqslant \left\{\frac{A^l + B^l}{2}\right\}^{1/l},$$

which proves the increasingness of $\{(A^k + B^k)/2\}^{1/k}$ along k. Since this sequence is bounded from above by $\{\lambda_1^{\cdot}(A) \vee \lambda_1^{\cdot}(B)\}I$, it has a limit, which will be denoted by $\Psi(A, B)$. It is readily seen that

$$\Psi(A, B) \geqslant A, B, \tag{6.54}$$

$$\Psi(A^k, B^k) = \Psi(A, B)^k, \qquad k = 1, 2, \dots; \tag{6.55}$$

hence

$$\Psi(A, B)^k \geqslant A^k, B^k, \qquad k = 1, 2, \dots. \tag{6.56}$$

We claim that $\Psi(A, B) \overset{\sigma}{\geqslant} A, B$. In view of (6.15), the inequality (6.56) implies

$$E(\Psi(A, B), r) \leqslant E(A, r), E(B, r) \qquad \text{for all} \quad r > 0. \tag{6.57}$$

Then, for any nondecreasing function $f(t)$ on R, integration by parts will show that, with sufficiently large α,

$$\begin{aligned} f(\Psi(A, B)) &= \int_0^\alpha f(r)\, dE(\Psi(A, B), r) \\ &= f(\alpha)I - \int_0^\alpha E(\Psi(A, B), r)\, df(r) \\ &\geqslant f(\alpha)I - \int_0^\alpha E(A, r)\, df(r) \qquad \text{by (6.57)} \\ &= f(A), \end{aligned}$$

and similarly

$$f(\Psi(A, B)) \geqslant f(B).$$

This establishes the claim. Next take any Hermitian C such that

$$C \overset{\sigma}{\geqslant} A \quad \text{and} \quad C \overset{\sigma}{\geqslant} B.$$

Use $f(t)=t^{kl}$, $k,l=1,2,\dots$, to see that

$$\frac{A^{kl}+B^{kl}}{2}\leqslant C^{kl},$$

and appeal to the monotone increasingness of the map $0\leqslant X\mapsto X^{1/k}$ to get $\Psi(A,B)^{l}\leqslant C^{l}$, $l=1,2,\dots$. Then $\Psi(A,B)\overset{\mathscr{S}}{\leqslant}C$ follows just as above. These observations show that $\Psi(A,B)$ is the supremum of the pair $\{A,B\}$ with respect to spectral order.

The second assertion results from the first by remarking that

$$A,B\overset{\mathscr{S}}{\geqslant}C\geqslant 0\quad\text{if and only if}\quad A^{-1},B^{-1}\overset{\mathscr{S}}{\leqslant}C^{-1}$$

as explained in the next section. ■

We are in position to present an eigenvalue analogue of (2.32) and (2.33).

THEOREM 6.16. *Let A, B be Hermitian. Then the following majorization relations hold*:

$$\vec{\lambda}^{\cdot}(A)\wedge\vec{\lambda}_{\cdot}(B)\underset{\cdot}{\prec}\vec{\lambda}(A\wedge B)\underset{\cdot}{\prec}\vec{\lambda}^{\cdot}(A)\wedge\vec{\lambda}^{\cdot}(B),\tag{6.58}$$

$$\vec{\lambda}^{\cdot}(A)\vee\vec{\lambda}_{\cdot}(B)\dot{\prec}\vec{\lambda}(A\vee B)\dot{\prec}\vec{\lambda}^{\cdot}(A)\vee\vec{\lambda}^{\cdot}(B),\tag{6.59}$$

where $A\vee B$ and $A\wedge B$ denote the supremum and the infimum of the pair A,B, respectively, with respect to spectral order.

Proof. In view of (6.52) we may assume that A,B are positive and invertible. First according to (6.39)

$$\frac{\vec{\lambda}^{\cdot}(A)^{k}+\vec{\lambda}_{\cdot}(B)^{k}}{2}\prec\vec{\lambda}\left(\frac{A^{k}+B^{k}}{2}\right)\prec\frac{\vec{\lambda}^{\cdot}(A)^{k}+\vec{\lambda}^{\cdot}(B)^{k}}{2},$$

and then, since $f(t):=-t^{1/k}$ is convex, by Corollary 2.2

$$\left\{\frac{\vec{\lambda}^{\cdot}(A)^{k}+\vec{\lambda}_{\cdot}(B)^{k}}{2}\right\}^{1/k}\dot{\prec}\vec{\lambda}\left(\left\{\frac{A^{k}+B^{k}}{2}\right\}^{1/k}\right)\dot{\prec}\left\{\frac{\vec{\lambda}^{\cdot}(A)^{k}+\vec{\lambda}^{\cdot}(B)^{k}}{2}\right\}^{1/k}.$$

Let $k\to\infty$ to reach (6.59). Finally (6.58) follows from (6.59) via (6.53). ■

NOTE. This section is largely based on the exposition of Amir-Moéz [6], Beckenbach and Bellman [7], Gohberg and Krein [32], Lidskii [44], Marshall and Olkin [51], and Markus [50]. The subdiagonalization (6.2)–(6.3) is in Schur [69]. Lemma 6.1 and Theorem 6.3 in their general form are due to Amir-Moéz [5], and in some special cases to Fan [23]. Corollary 6.4 is the Fischer-Courant theorem [24]. Theorem 6.7 is in Fan [22]. The observation (6.36) is due to Thompson [77]. Theorem 6.8 is due to Fan [22], Lidskii [43], and Wielandt [84] (see also Mirsky [54]); Theorem 6.9 is to Rotfeld [65] and Thompson [77]. Theorem 6.11 and Corollary 6.12 are the discovery of Weyl [82] (see also Fan [22]). A special case of Corollary 6.12 is in Schur [70]. Theorem 6.13 is due to Horn [36] and Lidskii [43]. Spectral order was considered by Olson [61]. The representation of the supremum in Lemma 6.15 is in Kato [38].

7. *Doubly Stochastic Maps*

In this section we develop an elementary part of the matrix versions of majorization. Denote by $\mathbf{M}_n$ the complex linear space of all complex n-square matrices, and by $\mathbf{H}_n$ its real subspace of Hermitian matrices. The space $\mathbf{M}_n$ is provided with a Hilbert-space structure

$$\langle A, B\rangle := \operatorname{tr}(B^*A) \qquad \text{for} \quad A, B \in \mathbf{M}_n, \tag{7.1}$$

and consequently

$$\|A\|^2 = \operatorname{tr}(A^*A). \tag{7.2}$$

Further $\mathbf{H}_n$ is provided with an order structure: $A \leqslant B$ if and only if $B - A$ is positive (semidefinite). Remark that the positive cone is *self-dual* in the sense

$$A \geqslant 0 \quad \text{if and only if} \quad \langle A, B\rangle \geqslant 0 \quad \text{for all} \;\; B \geqslant 0. \tag{7.3}$$

Since $\mathbf{M}_n$ and $\mathbf{M}_m$ are Hilbert spaces, with each linear map Φ from $\mathbf{M}_n$ to $\mathbf{M}_m$ is associated its *adjoint* Φ^* from $\mathbf{M}_m$ to $\mathbf{M}_n$:

$$\langle \Phi(A), B\rangle = \langle A, \Phi^*(B)\rangle \qquad \text{for} \quad A \in \mathbf{M}_n, \quad B \in \mathbf{M}_m. \tag{7.4}$$

A linear map Φ is said to be *positive* if

$$\Phi(A) \geqslant 0 \quad \text{whenever} \quad A \geqslant 0. \tag{7.5}$$

Φ is said to be *completely positive* if it admits a representation

$$\Phi(A) = \sum_{j=1}^{N} C_j^* A C_j \qquad \text{for all} \quad A \in \mathbf{M}_n, \tag{7.6}$$

where $C_1, \ldots, C_N$ are some $n \times m$ rectangular matrices. Complete positivity implies positivity, but not conversely. In view of (7.3) the adjoint of a positive linear map is positive. The adjoint of a completely positive map is completely positive. In fact, if Φ admits a representation (7.6), its adjoint has the following representation:

$$\Phi^*(B) = \sum_{j=1}^{N} C_j B C_j^* \qquad \text{for all} \quad B \in \mathbf{M}_m. \tag{7.7}$$

Let us introduce the majorization relation among Hermitian matrices in terms of their eigenvalue vectors. Recall first that, for each $A \in \mathbf{M}_n$, $\vec{\lambda}(A)$ is the n-vector of its eigenvalues, arranged in any order, with multiplicities counted. If A is Hermitian, then $\vec{\lambda}(A)$ is a real n-vector, and $\vec{\lambda}^{\cdot}(A)$ and $\vec{\lambda}_{\cdot}(A)$ are its decreasing and increasing rearrangements respectively. A Hermitian matrix A is said to be *majorized* by another B (in notation $A \prec B$) if $\vec{\lambda}(A) \prec \vec{\lambda}(B)$. *Submajorization* $A \underset{\cdot}{\prec} B$ and *supermajorization* $A \overset{\cdot}{\prec} B$ are defined correspondingly. Remark that the *equivalence* $A \sim B$, that is, simultaneous occurrence of $A \prec B$ and $A \succ B$, is nothing but unitary equivalence of A and B. Here note the following inequality:

$$|\langle A, B\rangle| \leqslant \sum_{j=1}^{n} \lambda^{\cdot}_j(|A|)\lambda^{\cdot}_j(|B|). \tag{7.8}$$

In fact,

$$\begin{aligned} |\langle A, B\rangle| \leqslant \operatorname{tr}\left|\vec{\lambda}(B^*A)\right| &\leqslant \sum_{j=1}^{n} \lambda^{\cdot}_j(|B^*A|) && \text{by (6.45)} \\ &\leqslant \sum_{j=1}^{n} \lambda^{\cdot}_j(|B^*|)\lambda^{\cdot}_j(|A|) && \text{by (6.49)} \\ &= \sum_{j=1}^{n} \lambda^{\cdot}_j(|A|)\lambda^{\cdot}_j(|B|). \end{aligned}$$

To formulate a noncommutative, i.e. matrix, analogue of Theorem 1.3, we need a notion of double stochasticity for a linear map. A linear map Φ on $\mathbf{M}_n$ is said to be *doubly stochastic* if it is positive, *unital* [i.e. $\Phi(I) = I$], and *trace-preserving* [i.e. $\mathrm{tr}(\Phi(A)) = \mathrm{tr}(A)$ for all $A \in \mathbf{M}_n$]. Remark that Φ is unital (trace-preserving) if and only if the adjoint Φ^* is trace-preserving (unital). Therefore the adjoint of a doubly stochastic map is doubly stochastic.

THEOREM 7.1. *The following conditions for Hermitian matrices $A, B \in \mathbf{H}_n$ are mutually equivalent:*

(i) $A \prec B$.
(ii) *There exist unitary matrices U_j and positive numbers $t_j > 0$ such that*

$$\sum_{j=1}^{N} t_j = 1 \quad \text{and} \quad A = \sum_{j=1}^{N} t_j U_j^* B U_j. \tag{7.9}$$

(iii) $A = \Phi(B)$ *for a completely positive, doubly stochastic map* Φ.
(iv) $A = \Phi(B)$ *for a doubly stochastic map* Φ.

Proof. (i) $\Rightarrow$ (ii): Since $\vec{\lambda}(A) \prec \vec{\lambda}(B)$ by definition, according to Theorem 1.3 there exist permutation matrices $\Pi^{(j)}$ and positive numbers t_j, $j = 1, \ldots, N$, such that

$$\vec{\lambda}(A) = \sum_{j=1}^{N} t_j \Pi^{(j)} \vec{\lambda}(B),$$

or equivalently

$$\mathrm{diag}\big(\vec{\lambda}(A)\big) = \sum_{j=1}^{N} t_j \Pi^{(j)*} \,\mathrm{diag}\big(\vec{\lambda}(B)\big)\, \Pi^{(j)}. \tag{7.10}$$

Take, according to (6.10), unitary matrices V, W such that

$$A = W^* \mathrm{diag}\big(\vec{\lambda}(A)\big) W \quad \text{and} \quad \mathrm{diag}\big(\vec{\lambda}(B)\big) = V^* B V. \tag{7.11}$$

Then $U_j := V \Pi^{(j)} W$ and $t_j > 0$, $j = 1, \ldots, N$, meet the requirement of (7.9), by (7.10).

(ii) $\Rightarrow$ (iii): Define a map Φ by

$$\Phi(X) := \sum_{j=1}^{N} t_j U_j^* X U_j \qquad \text{for} \quad X \in \mathbf{M}_n.$$

Then Φ is completely positive, doubly stochastic, and such that $\Phi(B) = A$.

(iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i): Suppose that $A = \Phi(B)$ for a doubly stochastic map Φ. Take unitary matrices V, W satisfying (7.11), and define a doubly stochastic map Ψ by

$$\Psi(X) := W\Phi(VXV^*)W^* \qquad \text{for} \quad X \in \mathbf{M}_n.$$

Then the assumption implies

$$\Psi\big(\operatorname{diag}(\vec{\lambda}(B))\big) = \operatorname{diag}(\vec{\lambda}(A)). \tag{7.12}$$

Let P_j be the orthogonal projection to the one-dimensional subspace spanned by

$$\vec{e}_{(j)} = (0,\dots,0,\overset{(j)}{1},0,\dots,0)^T.$$

Then (7.12) implies $\vec{\lambda}(A) = D\vec{\lambda}(B)$, where $D = (d_{ij})$ is an n-square matrix defined by $d_{ij} = \langle \Psi(P_j), P_i \rangle$ for all i and j. Evidently D is a doubly stochastic matrix, so that (i) follows from Theorem 1.3. ∎

Corollary 7.2. *If P_j, $j = 1,\dots,N$, are mutually annihilating orthogonal projections that sum to the identity:*

$$\sum_{j=1}^{N} P_j = I \quad \textit{and} \quad P_iP_j = \delta_{ij}P_j \quad \textit{for all } i \textit{ and } j, \tag{7.13}$$

then

$$\sum_{j=1}^{N} P_j A P_j \prec A \qquad \textit{for all Hermitian } A. \tag{7.14}$$

Proof. Since the matrix $U_j := 2P_j - I$ is unitary, the map

$$\Phi_j(X) := \tfrac{1}{2}X + \tfrac{1}{2}U_j^* X U_j$$

is doubly stochastic. Simple computation based on (7.13) will show

$$\sum_{j=1}^{N} P_j A P_j = \Phi_N \circ \cdots \circ \Phi_1(A),$$

where $\circ$ denotes composition. Since the composition of a finite number of doubly stochastic maps is again doubly stochastic, (7.14) follows from Theorem 7.1. ■

COROLLARY 7.3. *If A, B are positive n-square matrices, then the following majorizations hold in* $\mathbf{H}_{2n}$:

$$\begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix} \succ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \succ \begin{bmatrix} (A+B)/2 & 0 \\ 0 & (A+B)/2 \end{bmatrix}.$$

Proof. Since

$$\begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix},$$

the matrix on the left side is unitarily equivalent to

$$\begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix} \begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix},$$

which majorizes $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ by Corollary 7.2. This prove the left majorization. The right majorization results from Theorem 7.1 by observing that

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}$$

are unitarily equivalent. ■

Corollary 7.3 can be generalized, in a natural way, to the case of a finite number of positive matrices.

Since $\mathbf{H}_n$ and $\mathbf{H}_m$ are provided with the order structure $\leqslant$ and the majorization structure $\prec$, for a nonlinear map Φ from (a subset of) $\mathbf{H}_n$ to $\mathbf{H}_m$ we can speak about its *monotony*, *convexity*, *isotony*, *strong isotony*, etc., just as in Section 2. As Theorem 2.1 results from Theorem 1.3, the following statement results from Theorem 7.1: if Φ is convex and *unitarily invariant*, that is,

$$\Phi(U^*AU) = \Phi(A) \qquad \text{for all } A \text{ and all unitary } U,$$

then it is isotone. It becomes strong isotone if it is monotone increasing in addition.

Let $f(t)$ be a continuous function defined on the whole line $\mathbf{R}$ (or the half line $\mathbf{R}_+$). For each Hermitian (or positive) A, the matrix $f(A)$ is defined by (6.11). The map $A \mapsto f(A)$ is not necessarily convex even if $f(t)$ is convex. However, the map becomes isotone if $f(t)$ is convex. This follows via Theorem 2.1 from the relation $\vec{\lambda}(f(A)) = f(\vec{\lambda}(A))$. If, in addition, $f(t)$ is monotone increasing, the map is strongly isotone.

Here we take up the functions already used in Section 6, namely $1/t$ and t^p for $0 < p < 1$. The map $A \mapsto A^{-1}$ is monotone decreasing and convex on the set of positive invertible matrices. In fact, this follows from the identity

$$\langle A^{-1}\vec{x}, \vec{x}\rangle = \sup_{\vec{y}} \frac{|\langle \vec{x}, \vec{y}\rangle|^2}{\langle A\vec{y}, \vec{y}\rangle} \qquad \text{for all} \quad \vec{x} \in \mathbf{C}^n.$$

For $0 < p < 1$, the map $A \mapsto A^p$ is monotone increasing and concave. To see this, use the integral representation

$$s^p = \frac{\sin p\pi}{\pi} \int_0^\infty \frac{st^{p-1}}{s+t}\, dt$$

together with the spectral representation (6.13′) for A to get

$$A^p = \frac{\sin p\pi}{\pi} \int_0^\infty A(A+tI)^{-1} t^{p-1}\, dt$$

$$= \frac{\sin p\pi}{\pi} \int_0^\infty \left\{ I - t(A+tI)^{-1} \right\} t^{p-1}\, dt.$$

Since for each $t > 0$ the map $A \mapsto I - t(A+tI)^{-1}$ is monotone increasing and convex as proved above, so is the map $A \mapsto A^p$.

Nonlinear functionals on $\mathbf{H}_n$, that is, maps from $\mathbf{H}_n$ to $\mathbf{R}$, are of special importance. If $f(t)$ is convex, the functional $A \mapsto \operatorname{tr}(f(A))$ is isotone. Another example is this. To each Hermitian C, let us assign a function K_c defined by

$$K_c(A) = \sup_{U \text{ unitary}} \langle U^*AU, C\rangle. \tag{7.15}$$

K_c is unitarily invariant and convex, so that it is isotone.

THEOREM 7.4. *The following conditions for Hermitian matrices A, B are mutually equivalent*:

(i) $A \prec B$.
(ii) $K_c(A) \leqslant K_c(B)$ *for all Hermitian* C.
(iii) $\operatorname{tr}(f(A)) \leqslant \operatorname{tr}(f(B))$ *for all convex* f.
(iv) $\operatorname{tr}|A - tI| \leqslant \operatorname{tr}|B - tI|$ *for all* $t \in \mathbf{R}$.

Proof. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are immediate.

(ii) $\Rightarrow$ (i): Let C be any orthogonal projection of rank k. Then in view of Theorem 6.3

$$K_c(X) = \sum_{j=1}^{k} \lambda_j^{\cdot}(X) \qquad \text{for all} \quad X \in \mathbf{H}_n.$$

Therefore (ii) implies $\vec{\lambda}(A) \underset{\cdot}{\prec} \vec{\lambda}(B)$. Use $C = I$ and $= -I$ to see that $\operatorname{tr}(\vec{\lambda}(A)) = \operatorname{tr}(\vec{\lambda}(B))$.

(iv) $\Rightarrow$ (i): Condition (iv) is equivalent to

$$\operatorname{tr}\left|\vec{\lambda}(A) - t\vec{e}\right| \leqslant \operatorname{tr}\left|\vec{\lambda}(B) - t\vec{e}\right| \qquad \text{for all} \quad t \in \mathbf{R},$$

and Corollary 1.2 can be applied. ∎

Let $\Phi(\vec{x})$ be a seminorm on $\mathbf{C}^n$, that is,

(*absolute homogeneity*): $\Phi(\alpha\vec{x}) = |\alpha|\Phi(\vec{x}) \quad \text{for} \quad \alpha \in \mathbf{C}, \qquad (7.16)$

(*subadditivity*): $\Phi(\vec{x} + \vec{y}) \leqslant \Phi(\vec{x}) + \Phi(\vec{y}). \qquad (7.17)$

If $\Phi(\vec{x})$ is an *absolute* seminorm, that is,

$$\Phi(\vec{x}) = \Phi(|\vec{x}|) \qquad \text{for all} \quad \vec{x} \in \mathbf{C}^n, \tag{7.18}$$

then it is monotone increasing on the set of positive vectors. In fact, if $0 \leqslant \vec{x} \leqslant \vec{y}$, then $\vec{x}$ is a convex combination of vectors whose moduli coincide either with $\vec{y}$ or 0. A permutation-invariant, absolute seminorm is called a *symmetric gauge function*. To each such function Φ on $\mathbf{C}^n$, let us assign a seminorm $\|\cdot\|_\Phi$ on $\mathbf{M}_n$ by

$$\|A\|_\Phi := \Phi\big(\vec{\lambda}(|A|)\big) \qquad \text{for} \quad A \in \mathbf{M}_n. \tag{7.19}$$

Absolute homogeneity follows from (7.16). According to (6.40)

$$\vec{\lambda}(|A+B|) \prec_{w} \vec{\lambda}^{\cdot}(|A|) + \vec{\lambda}^{\cdot}(|B|), \tag{7.20}$$

and by Corollary 2.3 $\Phi(\vec{x})$ is strongly isotone. Thus subadditivity of $\|\cdot\|_\Phi$ results from (7.20) via (7.17). This seminorm is *unitarily invariant* in the sense

$$\|UAV\|_\Phi = \|A\|_\Phi \qquad \text{for all unitary } U, V. \tag{7.21}$$

Conversely, any unitarily invariant seminorm on $\mathbf{M}_n$ is obtained in this way.

The Hilbert-space norm $\|A\|$ on $\mathbf{M}_n$ corresponds to $\Phi(\vec{x}) = \{\sum_{j=1}^n |x_j|^2\}^{1/2}$. The norm that corresponds to $\Phi(\vec{x}) = \max_{1 \leqslant j \leqslant n} |x_j|$ is called the *spectral norm* or *operator norm* and denoted by $\|\cdot\|_\infty$;

$$\|A\|_\infty := \lambda_1^{\cdot}(|A|) \qquad \text{for} \quad A \in \mathbf{M}_n. \tag{7.22}$$

$\mathbf{M}_n$ becomes a C^*-algebra with respect to the spectral norm, that is,

$$\|AB\|_\infty \leqslant \|A\|_\infty \cdot \|B\|_\infty, \qquad \|A^*A\|_\infty = \|A\|_\infty^2, \quad \text{and} \quad \|I\|_\infty = 1. \tag{7.23}$$

When $\Phi(\vec{x}) = \sum_{j=1}^n |x_j|$, the corresponding norm is called the *trace norm* and is denoted by $\|\cdot\|_1$;

$$\|A\|_1 := \sum_{j=1}^{n} \lambda_j(|A|) = \operatorname{tr}|A|. \tag{7.24}$$

An important fact is that the spectral norm and the trace norm are dual to each other in the sense that

$$\|A\|_1 = \sup_{B \neq 0} \frac{|\langle A, B\rangle|}{\|B\|_\infty} \qquad \text{for all} \quad A \in \mathbf{M}_n \tag{7.25}$$

and

$$\|A\|_\infty = \sup_{B \neq 0} \frac{|\langle A, B\rangle|}{\|B\|_1} \qquad \text{for all} \quad A \in \mathbf{M}_n. \tag{7.26}$$

This follows from (7.8).

Let us turn to concave functions.

THEOREM 7.5. *If $f(t)$ is nonnegative and concave on the half line $\mathbf{R}_+$ and $f(0) = 0$, the functional $\|\cdot\|_f$ on $\mathbf{M}_n$ defined by*

$$\|A\|_f := \sum_{j=1}^{n} f(\lambda_j(|A|)) \qquad \text{for} \quad A \in \mathbf{M}_n, \tag{7.27}$$

is subadditive.

Proof. The assumption implies that $f(t)$ is monotone increasing and subadditive on $\mathbf{R}_+$. According to (6.36), for $A, B \in \mathbf{M}_n$, there exist unitary matrices U, V such that $|A + B| \leqslant U^*|A|U + V^*|B|V$; hence by (6.31)

$$\|A + B\|_f \leqslant \|U^*|A|U + V^*|B|V\|_f. \tag{7.28}$$

The functional Φ on $\mathbf{C}^{2n} = \mathbf{C}^n \oplus \mathbf{C}^n$ defined by

$$\Phi(\vec{x}, \vec{y}) = \sum_{j=1}^{n} f(|x_j|) + \sum_{j=1}^{n} f(|y_j|) \tag{7.29}$$

is permutation-invariant and concave on the set of positive vectors, so that in view of Corollary 2.3, $-\Phi$ is isotone. On the other hand, by Corollary 7.3

$$\begin{aligned}\left(\vec{\lambda}(|A|), \vec{\lambda}(|B|)\right) &= \left(\vec{\lambda}(U^*|A|U), \vec{\lambda}(V^*|B|V)\right)\\ &\prec \left(\vec{\lambda}(U^*|A|U + V^*|B|V), 0\right);\end{aligned}$$

hence by (7.27) and (7.29)

$$\|U^*|A|U+V^*|B|V\|_f \leqslant \|A\|_f + \|B\|_f. \tag{7.30}$$

Now the expected subadditivity follows from (7.28) and (7.30). ∎

Take $f(t)=t/(1+t)$, for example. Then Theorem 7.5 tells us

$$\sum_{j=1}^{n} \frac{\lambda_j(|A+B|)}{1+\lambda_j(|A+B|)} \leqslant \sum_{j=1}^{n} \frac{\lambda_j(|A|)}{1+\lambda_j(|A|)} + \sum_{j=1}^{n} \frac{\lambda_j(|B|)}{1+\lambda_j(|B|)}. \tag{7.31}$$

It follows from Theorem 7.1 via (2.8) that if Φ is a doubly stochastic map and A is Hermitian, then $|\Phi(A)| \underset{\cdot}{\prec} |A|$. To generalize this to general matrices, we start with a lemma.

Lemma 7.6. *If Φ is a doubly stochastic map, then*

$$\|\Phi(A)\|_\infty \leqslant \|A\|_\infty \quad \textit{and} \quad \|\Phi(A)\|_1 \leqslant \|A\|_1 \qquad \textit{for all} \quad A \in \mathbf{M}_n. \tag{7.32}$$

Proof. Let us prove first that $\|\Phi(U)\|_\infty \leqslant 1$ for any unitary U. We know that U admits a representation $U=\sum_{j=1}^n \zeta_j P_j$ where P_j are mutually annihilating orthogonal projections that sum to I, and $|\zeta_j|=1$, $j=1,\dots,n$. Take normalized vectors $\vec{x}, \vec{y}$ such that $\langle \Phi(U)\vec{x}, \vec{y}\rangle = \|\Phi(U)\|_\infty$. Then

$$\|\Phi(U)\|_\infty \leqslant \sum_{j=1}^{n} \left|\left\langle \Phi(P_j)\vec{x}, \vec{y}\right\rangle\right|$$

$$\leqslant \sum_{j=1}^{n} \left\langle \Phi(P_j)\vec{x}, \vec{x}\right\rangle^{1/2} \left\langle \Phi(P_j)\vec{y}, \vec{y}\right\rangle^{1/2}$$

by the Schwarz inequality

$$\leqslant \left\{ \sum_{j=1}^{n} \left\langle \Phi(P_j)\vec{x}, \vec{x}\right\rangle \right\}^{1/2} \left\{ \sum_{j=1}^{n} \left\langle \Phi(P_j)\vec{y}, \vec{y}\right\rangle \right\}^{1/2}$$

by the Cauchy inequality

$$= \|\vec{x}\| \cdot \|\vec{y}\| = 1 \qquad (\Phi \text{ is unital}).$$

Now any $A \in \mathbf{M}_n$ with $\|A\|_\infty \leqslant 1$ is a convex combination of unitary matrices. In fact, first represent $A = U\operatorname{diag}(\vec{\lambda}(|A|))V$ with some unitary matrices U, V and use the arguments in the proof of Corollary 4.4. Then it follows that $\|\Phi(A)\|_\infty \leqslant 1$ whenever $\|A\|_\infty \leqslant 1$, which establishes the left inequality of (7.32). Apply the same arguments to Φ^* to see that $\|\Phi^*(A)\|_\infty \leqslant \|A\|_\infty$ for all $A \in \mathbf{M}_n$, which implies the right inequality of (7.32) via (7.25). ∎

THEOREM 7.7. *If Φ is a doubly stochastic map on $\mathbf{M}_n$, then*

$$|\Phi(A)| \underset{\cdot}{\prec} |A| \qquad \text{for all} \quad A \in \mathbf{M}_n. \tag{7.33}$$

Proof. First let us show that $|\Phi^*(UP)| \underset{\cdot}{\prec} |UP|$ whenever U is unitary and P is an orthogonal projection (of rank k). In fact, since Φ^* is doubly stochastic, by Lemma 7.6

$$\sum_{j=1}^{n} \lambda_j^{\cdot}(|\Phi^*(UP)|) = \|\Phi^*(UP)\|_1 \leqslant \|UP\|_1 = \|P\|_1 = k$$

and

$$\lambda_1^{\cdot}(|\Phi^*(UP)|) = \|\Phi^*(UP)\|_\infty \leqslant \|UP\|_\infty = \|P\|_\infty = 1,$$

which implies

$$\vec{\lambda}(|\Phi^*(UP)|) \underset{\cdot}{\prec} \vec{\lambda}(|UP|) = (\overbrace{1,\dots,1}^{k},0,\dots,0)^T. \tag{7.34}$$

Now for any $A \in \mathbf{M}_n$ and $k \leqslant n$ there exist a unitary matrix U and an orthogonal projection P of rank k such that

$$\sum_{j=1}^{k} \lambda_j^{\cdot}(|\Phi(A)|) = \langle UP, \Phi(A) \rangle.$$

Then, with $\lambda^{\cdot}_{n+1}(|A|)=0$,

$$|\langle UP,\Phi(A)\rangle| = |\langle \Phi^*(UP),A\rangle| \leqslant \sum_{j=1}^{n} \lambda^{\cdot}_j(|A|)\lambda^{\cdot}_j(|\Phi^*(UP)|) \qquad \text{by (7.8)}$$

$$= \sum_{j=1}^{n} \{\lambda^{\cdot}_j(|A|)-\lambda^{\cdot}_{j+1}(|A|)\} \sum_{i=1}^{j} \lambda^{\cdot}_i(|\Phi^*(UP)|)$$

$$\leqslant \sum_{j=1}^{n} \{\lambda^{\cdot}_j(|A|)-\lambda^{\cdot}_{j+1}(|A|)\} \sum_{i=1}^{j} \lambda^{\cdot}_i(|UP|) \qquad \text{by (7.34)}$$

$$= \sum_{j=1}^{k} \lambda^{\cdot}_j(|A|),$$

which proves (7.33). ∎

COROLLARY 7.8. *If Φ is a doubly stochastic map on $\mathbf{M}_n$, for any unitarily invariant norm $\|\cdot\|_\#$*

$$\|\Phi(A)\|_\# \leqslant \|A\|_\# \qquad \text{for all} \quad A\in\mathbf{M}_n.$$

NOTE. Majorization in the dual space of a C^*-algebra—in particular, in a matrix algebra—is the subject of Alberti and Uhlmann [2, 3, 80, 81]; Grothendieck [33] considered a von Neumann algebra with trace. Positive linear maps have been discussed in the framework of C^*-algebra (see [19] and [76]). Complete positivity was introduced by Stinespring [75]. Its equivalence to (7.6) for the case of matrix algebras is due to Choi [12]. Corollary 7.2 is in Fan [23]. Corollary 7.3 was proved by Thompson [77]. When the map $A\mapsto f(A)$ generated by a function f becomes monotone or convex is the subject of the classical papers of Löwner [48] and Krauss [41] (see [20] for exposition). Symmetric gauge functions were introduced by von Neumann [59], who studied metrizations of matrix spaces. See [32] and [54] for exposition. Theorem 7.5 is due to Rotfeld [65], but the proof in the paper is due to Thompson [77]. Thompson [78] studied in detail triangular-type inequalities among matrices.

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