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On Tamari lattices

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Abstract

Tamari lattices are defined as the set of all binary bracketings on a fixed number of symbols ordered by applying the associativity rule only in one direction. Using methods of formal concept analysis we derive a recursive construction of these lattices based on successive doublings of intervals. It turns out that for every $n \in \mathbb{N}$ Tamari lattices and their congruence lattices have the same number of elements and the same number of coverings, both connected with the Catalan numbers.

1. Introduction

The purpose of this paper is to study the structure of Tamari lattices. These lattices are defined as the set of all binary bracketings on $n+1$ symbols ordered by applying the associativity rule only in one direction. The nontrivial proof that this order constitutes indeed a lattice uses a vector representation of the lattice elements (cf. [7]). Recently, Bennett and Birkhoff investigated these lattices (see [1]) and gave a description of their irreducibles. In [8], Markowsky has stated some properties of Tamari lattices concerning complementation, minimal and maximal chains, and retracts. It seems that they did not know the substantial paper of Urquhart [13].

Our study is based on the formal context corresponding to a Tamari lattice. Basic definitions and results of formal concept analysis are stated in this introduction or later when they are needed. Section 2 contains results of Bennett and Birkhoff which lead to a recursive description of the appropriate contexts. In Section 3, we introduce the arrow relations of a context which provide an easy proof that Tamari lattices are bounded subdirectly irreducible lattices. By a result of Day, bounded lattices can be generated from the two-element lattice by applying the interval doubling construction. A corresponding context construction was given in [6]. Thus, the recursive description of the underlying contexts yields to a construction method for Tamari

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lattices. As an example, the appendix contains the construction of T_5 with 42 elements starting with T_4. The arrow graph defined in Section 4 yields the result that, for each $n \in \mathbb{N}$, Tamari lattices and their congruence lattices have the same number of elements. Tamari lattices are naturally connected with the Catalan numbers C_n. The last section proves that also the number of coverings in both, the Tamari lattice T_n and the corresponding congruence lattice, equals $((n-1)/2)\ C_n$.

Let us recall some basic definitions of lattice theory and formal concept analysis. For a finite lattice L, the set of all join-irreducible elements is denoted by $J(L)$, the set of all meet-irreducible elements of L by $M(L)$. A set $S \subseteq L$ is *supremum-dense* (*infimum-dense*) if every lattice element $x \in L$ can be represented by $x = \bigvee S_0$ ($x = \bigwedge S_0$) with $S_0 \subseteq S$. A *context* $\mathbb{K}$ is given by $\mathbb{K} := (G, M, I)$, where G and M are sets and $I \subseteq G \times M$ is a binary relation. The elements of G and M are called *objects* and *attributes*, respectively. A context can be represented by a cross-table with one row for each object and one column for each attribute; there is a cross in row g and column m if and only if gIm. The context $\mathbb{K}$ determines the *concept lattice* of $\mathbb{K}$ which we denote by $\mathfrak{B}(\mathbb{K})$. For a finite lattice L, we get $L \cong \mathfrak{B}(\mathbb{K})$ with $\mathbb{K} := (G, M, \leqslant)$ if $G \subseteq L$ is supremum-dense and $M \subseteq L$ is infimum-dense in L. In this case, L can be reconstructed from $\mathbb{K}$. $\mathbb{K}$ is called *reduced* if the sets G and M are minimal with respect to these properties. For a finite lattice L, there is up to isomorphism only one reduced context $\mathbb{K}(L) := (J(L), M(L), \leqslant)$.

2. Definition of Tamari lattices

In this section we introduce Tamari lattices by an order defined on the set of all possible bracketings on a fixed number of symbols. An alternative definition by integer-valued vectors satisfying two conditions was given by Huang and Tamari (cf. [7]). In [1], Bennett and Birkhoff investigate these lattices and determine their irreducibles and the comparability relation between them. These results lead to a recursive definition of the appropriate contexts. We start with the original definition:

Definition 2.1. For each $n \in \mathbb{N}$, the elements of the *Tamari lattice* T_n are all binary bracketings on $n+1$ fixed symbols, say $x_0, \ldots, x_n$. They can be ordered by the following semi-associativity rule:

(SA) $\qquad (AB)C \to A(BC)$.

For $t_1, t_2 \in T_n$, $t_1 \leqslant t_2$ if and only if t_1 can be transformed into t_2 by (repeated) application of (SA).

Table 1 contains different descriptions of the 5 elements of the Tamari Lattice T_3 which is isomorphic to the nonmodular lattice N_5. All possible binary bracketings on the four symbols x_0, x_1, x_2, and x_3 are listed in the first column.

Table 1
The elements of T_n

Binary bracketing	Right bracketing	3-vectors	$J(T_3)$	$M(T_3)$
$(x_0(x_1(x_2x_3)))$	$x_0(x_1(x_2(x_3)))$	(3, 3, 3)	—	—
$(x_0((x_1x_2)x_3))$	$x_0(x_1(x_2)(x_3))$	(3, 2, 3)	[1, 3]	⟨2, 2⟩
$((x_0(x_1x_2))x_3)$	$x_0(x_1(x_2))(x_3)$	(2, 2, 3)	[1, 2]	⟨1, 2⟩
$((x_0x_1)(x_2x_3))$	$x_0(x_1)(x_2(x_3))$	(1, 3, 3)	[2, 3]	⟨1, 1⟩
$(((x_0x_1)x_2)x_3)$	$x_0(x_1)(x_2)(x_3)$	(1, 2, 3)	—	—

Huang and Tamari introduce the *right bracketing* convection to prove the lattice property for T_n. Notice that the distribution of the closing brackets uniquely determines the binary bracketing. In the corresponding right bracketing we choose a constant distribution of the opening brackets just one before each symbol $x_i (i=1,\ldots,n)$ while the closing ones are left fixed. The second column in Table 1 contains the corresponding right bracketing.

Each right bracketing can be encoded by an n-vector $v=(v_1,\ldots,v_n)$ with $v_i \in \{1,\ldots,n\} (i=1,\ldots,n)$. $v_i=j$ if and only if the opening bracket before x_i closes after x_j. Of course, no closing bracket occurs before the corresponding opening one. Hence, $v_i \geqslant i$ for $i=1,\ldots,n$. Furthermore, two different pairs of brackets do not overlap. This can be captured by the condition that $v_i \geqslant j > i$ implies $v_i \geqslant v_j$ for $i=1,\ldots,n$. Indeed, these two conditions characterize all possible n-vectors corresponding to right bracketings. The following proposition can be found in [7].

Proposition 2.2. *For each $n \in \mathbb{N}$, the lattice T_n is isomorphic to the set of all n-vectors $(v_1,\ldots,v_n)$ of positive integers $\leqslant n$ satisfying the following properties:*
(1) *$i \leqslant v_i$ for $i=1,\ldots,n$;*
(2) *$i \leqslant j \leqslant v_i$ implies $v_j \leqslant v_i$ for $i=1,\ldots,n$.*
The order in this set is defined by componentwise comparison.

From now on, we will identify T_n with the lattice of these n-vectors. The third column in Table 1 contains the appropriate n-vectors. The meet in T_n is computed componentwise. In [8], Markowsky describes an algorithm to compute joins in T_n. For $x, y \in T_n$, let $z := x \vee y$. First, compute the componentwise join w by $w_k := x_k \vee y_k$ $(k=1,\ldots,n)$. This gives not always a lattice element. For $k=n,\ldots,1$, we have to increase the entries recursively by $z_k := \max(\{w_k\} \cup \{z_j \mid j=k+1,\ldots,w_k\})$. In T_3, we have to apply this increasing procedure for the join of (2, 2, 3) and (1, 3, 3). In T_n, the bottom element is $0 := (1, 2, \ldots, n)$ and the top element is given by $1 := (n, n, \ldots, n)$. Next, we will determine the irreducibles in T_n. The following proposition can be found in the paper of Bennett and Birkhoff (cf. [1]).

Proposition 2.3. (1) *The join-irreducibles of T_n are of the form*

$$[j,k] := (1,\dots,j-1,k,j+1,\dots,n)$$

with $j<k$ and $j<n$. So $[j,k]_l := j$ if $l \neq j$ and $[j,k]_j := k$.

(2) *The meet-irreducibles of T_n are of the form*

$$\langle p,q\rangle := (n,\dots,n,q,\dots,q,n,\dots,n)$$

with $p \leqslant q$ and $q<n$. So $\langle p,q\rangle_l := q$ if $p \leqslant l \leqslant q$ and $\langle p,q\rangle_l := n$ otherwise.

Proof. Let $v \in T_n$. For every entry v_i with $v_i > i$ we have $v \geqslant [i,v_i]$. Hence, $v = \bigvee\{[i,v_i] \mid v_i > i\}$. Thus, v is join-reducible if there is more than one entry v_i with $v_i > i$. On the other hand, if $v = [j,k]$ then every join representation of $[j,k]$ contains an element $w \in T_n$ with $w_j = k$. But now $w \geqslant [j,k]$, so $[j,k]$ is join-irreducible. This proves (1).

For (2), observe that for every entry $v_j < n$ there is a least index i with $v_i = v_j < n$. Thus, we have $v \leqslant \langle i, v_i\rangle$. We obtain $v = \bigwedge\{\langle i,v_i\rangle \mid v_i < n \text{ and } v_i \neq v_{i-1}\}$. So every meet-irreducible element must be of the form $\langle p,q\rangle$. Now, consider a meet representation of $\langle p,q\rangle$. This representation must contain an element w with $w_p = q$ which yields $\langle p,q\rangle \geqslant w$. So $\langle p,q\rangle$ is meet-irreducible. □

The last two columns of Table 1 contain the join- and meet-irreducibles of T_3. The comparability between them can be easily derived (cf. [1]).

Proposition 2.4. *Let $[j,k] \in J(T_n)$ and $\langle p,q\rangle \in M(T_n)$. Then $[j,k] \nleqslant \langle p,q\rangle \Leftrightarrow p \leqslant j \leqslant q < k$.*

Proof. Consider the following equivalences:

$$\begin{aligned}[j,k] \nleqslant \langle p,q\rangle &\Leftrightarrow [j,k]_j \nleqslant \langle p,q\rangle_j \\ &\Leftrightarrow k \nleqslant q \text{ and } p \leqslant j \leqslant q \\ &\Leftrightarrow p \leqslant j \leqslant q < k. \qquad \square\end{aligned}$$

Now, we can state the main theorem of this section:

Theorem 2.5. *Let $\leqslant_n$ denote the order in T_n and let $\mathbb{K}(T_n) := (J(T_n), M(T_n), \leqslant_n)$ be the canonical context for T_n. Then $\mathbb{K}(T_{n+1}) := (J(T_{n+1}), M(T_{n+1}), \leqslant_{n+1})$ can be constructed recursively from $\mathbb{K}(T_n)$ by*

$$J(T_{n+1}) := J(T_n) \cup \{[l,n+1] \mid l = 1,\dots,n\},$$

$$M(T_{n+1}) := M(T_n) \cup \{\langle l,n\rangle \mid l = 1,\dots,n\},$$

and the following incidence relation:

$$[j,k] \leqslant_{n+1} \langle p,q\rangle \ :\Leftrightarrow \begin{cases} k\leqslant n, q\leqslant n-1, \text{ and } [j,k]\leqslant_n \langle p,q\rangle, \text{ or} \\ k\leqslant n, q=n, \text{ or} \\ k=n+1,\ q\leqslant n-1, \text{ and } j=n, \text{ or} \\ k=n+1,\ q\leqslant n-1, \text{ and } [j,k-1]\leqslant_n \langle p,q\rangle, \text{ or} \\ k=n+1,\ q=n, \text{ and } j<p. \end{cases}$$

Proof. First of all notice that the elements of $J(T_n) \subseteq J(T_{n+1})$ and $M(T_n) \subseteq M(T_{n+1})$ describe different vectors in T_n and in T_{n+1}, respectively. Nevertheless, the order relation remains unchanged because the definition of noncomparability does not refer to n or $n+1$, respectively. Now, we have to consider the four other cases in the definition of $\leqslant_{n+1}$. If $k\leqslant n$ and $q=n$ then $k\leqslant n=q$ so there are always incidences. From now on, assume $k=n+1$. The definition of noncomparability reduces to $p\leqslant j\leqslant q$. The case $q\leqslant n-1$ and $j=n$ yields that the corresponding elements are comparable. If $q\leqslant n-1$ and $j<n$ then $p\leqslant j\leqslant q<n+1$ and $p\leqslant j\leqslant q<n$ are equivalent. Hence, $[j,n+1]\leqslant_{n+1} \langle p,q\rangle$ if and only if $[j,n]\leqslant_n \langle p,q\rangle$. Now, observe that in case of $k=n+1$ and $q=n$ noncomparability reduces to $p\leqslant j$. Thus, $[j,n+1]\leqslant_{n+1} \langle p,n\rangle$ if and only if $j<p$. $\square$

In the language of cross-tables the construction works as follows: Add n rows and n columns to the cross-table representing $\mathbb{K}(T_n)$. One gets four rectangles: Then one on the top left represents the cross-table of $\mathbb{K}(T_n)$, the one on the top right has to be filled with crosses. In the rectangle on the bottom left, the first row is filled with crosses while the other $n-1$ rows contain a copy of the last $n-1$ rows of $\mathbb{K}(T_n)$. The rectangle on the bottom right is filled with crosses only strictly below the diagonal. Fig. 1 contains $\mathbb{K}(T_5)$ where the double lines indicate the several steps of this iteration process (the arrows are explained in the next section).

	$\langle 1,1\rangle$	$\langle 2,2\rangle$	$\langle 1,2\rangle$	$\langle 3,3\rangle$	$\langle 2,3\rangle$	$\langle 1,3\rangle$	$\langle 4,4\rangle$	$\langle 3,4\rangle$	$\langle 2,4\rangle$	$\langle 1,4\rangle$
$[1,2]$	⤢	×	×	×	×	×	×	×	×	×
$[2,3]$	×	⤢	↙	×	×	×	×	×	×	×
$[1,3]$	↗	×	⤢	×	×	×	×	×	×	×
$[3,4]$	×	×	×	⤢	↙	↙	×	×	×	×
$[2,4]$	×	↗		×	⤢	↙	×	×	×	×
$[1,4]$	↗	×	↗	×	×	⤢	×	×	×	×
$[4,5]$	×	×	×	×	×	×	⤢	↙	↙	↙
$[3,5]$	×	×	×	↗			×	⤢	↙	↙
$[2,5]$	×	↗		×	↗		×	×	⤢	↙
$[1,5]$	↗	×	↗	×	×	↗	×	×	×	⤢

Fig. 1. $\mathbb{K}(T_5)$.

3. Bounded lattices and the doubling construction

Using the recursive definition of $\mathbb{K}(T_n)$ in the last section we can deduce some structural informations about Tamari lattices. We recall the following definitions from [15].

Definition 3.1. For $g \in J(L)$ we define the *intent* g' of g by

$$g' = \{m \in M(L) \mid g \leqslant m\}.$$

Dually, the *extent* m' of an element $m \in M(L)$ is defined by

$$m' = \{g \in J(L) \mid g \leqslant m\}.$$

These sets can be ordered by set inclusion. Now, for $g \in J(L)$ and $m \in M(L)$, we define arrow relations by

$$g \nearrow m \;:\Leftrightarrow\; g \nleqslant m \text{ and } m' \text{ is maximal in } \{n' \mid n \in M(L) \text{ and } g \nleqslant n\},$$

$$g \swarrow m \;:\Leftrightarrow\; g \nleqslant m \text{ and } g' \text{ is maximal in } \{h' \mid h \in J(L) \text{ and } h \nleqslant m\},$$

$$g \swarrow\!\!\!\!\nearrow m \;:\Leftrightarrow\; g \nearrow m \text{ and } g \swarrow m.$$

These sets are called the *arrow-up relation*, *arrow-down relation*, and *double-arrow relation*, respectively. As usual, we denote them by $\swarrow$, $\nearrow$, and $\swarrow\!\!\!\!\nearrow$, respectively.

In the next lemma, we compute these relations in case of $\mathbb{K}(T_n)$.

Lemma 3.2. *Let $\mathbb{K}(T_n)$ be the context of the Tamari lattice T_n given by the recursive description in the last section. Let $g \in J(T_n)$ and $m \in M(T_n)$. Then*

$$g \nearrow m \text{ in } \mathbb{K}(T_n) := \begin{cases} g \in J(T_{n-1}),\ m \in M(T_{n-1}) \text{ and } g \nearrow m \text{ in } \mathbb{K}(T_{n-1}), \text{ or} \\ g = [j, n+1],\ m \in M(T_{n-1}) \text{ and } [j, n] \nearrow m \text{ in } \mathbb{K}(T_{n-1}), \text{ or} \\ g = [j,\ n+1] \text{ and } m = \langle j, n \rangle (j = 1, \dots, n). \end{cases}$$

$$g \swarrow m \text{ in } \mathbb{K}(T_n) := \begin{cases} g \in J(T_{n-1}),\ m \in M(T_{n-1}) \text{ and } g \swarrow m \text{ in } \mathbb{K}(T_{n-1}), \text{ or} \\ g = [j,\ n+1],\ m = \langle p, n \rangle \text{ and } p \leqslant j. \end{cases}$$

Proof. We consider $g \in J(T_n)$ and $m \in M(T_n)$ with $g \nleqslant m$. First, we will check the arrow-up relation: Let $m := \langle p, q \rangle \in M(T_{n-1})$ and $m_1 := \langle p_1, n \rangle \notin M(T_{n-1})$. Then $[n, n+1] \leqslant m$ but $[n, n+1] \nleqslant m_1$; *so* m' cannot be dominated by m_1' with $m_1 \notin M(T_{n-1})$. We have $[j, n+1]' \cap M(T_{n-1}) = [j, n]' \cap M(T_{n-1})$ for $j = 1, \dots, n-1$. This together with $[n, n+1]' = M(T_{n-1})$ yields that the order relation within the set $\{m' \mid m \in M(T_{n-1})\}$ remains the same. Thus, if $g \in J(T_{n-1})$ then $g \nearrow m$ in $\mathbb{K}(T_{n-1})$ if and

only if $g \nearrow m$ $\mathbb{K}(T_n)$. Moreover, for $g=[j,n+1]$ we have $g \nearrow m$ in $\mathbb{K}(T_n)$ if and only if $[j,n] \nearrow m$ in $\mathbb{K}(T_{n-1})$ by the remark above. If $m:=\langle p,n\rangle \notin M(T_{n-1})$ then m' cannot be dominated by m_1' with $m_1:=\langle p_1,q_1\rangle \in M(T_{n-1})$ because $[p_1,q_1+1]\in\langle p,n\rangle'$ but $[p_1,q_1+1]\notin\langle p_1,q_1\rangle'$. Now observe that $\langle n,n\rangle' \supset \langle n-1,n\rangle' \supset \cdots \supset \langle 1,n\rangle'$. Hence, for $g=[j,n+1]$ we have $[j,n+1] \nearrow \langle p,n\rangle$ if and only if $j=p$. (If $j<p$ then $[j,n+1]\leqslant\langle p,n\rangle$ and if $j>p$ then $j\geqslant p+1$ and $\langle p+1,n\rangle' \supset \langle p,n\rangle'$).

Next, we consider the arrow-down relation in $\mathbb{K}(T_n)$. Let $g:=[j,k]\in J(T_{n-1})$. Then g' cannot be dominated by h' with $h\notin J(T_{n-1})$ because $g\leqslant\langle 1,n\rangle$ and $h \nleqslant \langle 1,n\rangle$. We have $g'\cap(M(T_n)\setminus M(T_{n-1}))=h'\cap(M(T_n)\setminus M(T_{n-1}))$ for $g,h\in J(T_{n-1})$. So the induced order within $J(T_{n-1})$ remains unchanged. Thus, we get $g \swarrow m$ in $\mathbb{K}(T_n)$ if and only if $g \swarrow m$ *in* $\mathbb{K}(T_{n-1})$. If $g:=[j,n+1]\notin J(T_{n-1})$ then g' is dominated by $[j,n]'$. Hence, $g \swarrow m$ if and only if $m\notin M(T_{n-1})$. Now, we remark that $[j,n+1]\leqslant\langle p,p\rangle$ if and only if $j\neq p$. Thus, for $j_1\neq j_2$, the intents $[j_1,n+1]'$ and $[j_2,n+1]'$ are incomparable. Let $g:=[j,n+1]$ and $m:=\langle p,n\rangle$ with $p\leqslant j$. The observation above together with the fact that $h\leqslant\langle p,n\rangle$ for $h\in J(T_{n-1})$ implies that g' is maximal in $\{h' \mid h\in J(T_n)$ and $h \nleqslant m\}$. Hence, $g \swarrow m$ in this case. □

In the language of cross-tables we can establish the following description: in the cross-table representing $\mathbb{K}(T_n)$ we can fill an arrow up (down) into the cell determined by row g and column m if $g \nearrow m$ ($g \swarrow m$). This can be done because the arrow relation and the $\leqslant$ relation are disjoint. Then we get double-arrows exactly on the diagonal; the other empty cells of the $1\times 1, 2\times 2, \ldots, (n-1)\times(n-1)$ diagonal blocks are filled with arrows down. Arrows up are only below the diagonal; they can be copied in the recursion procedure.

We will use this description of the arrow structure of $\mathbb{K}(T_n)$ to prove that Tamari lattices are bounded subdirectly irreducible lattices. We repeat the definitions.

Definition 3.3. A finite lattice L is *bounded* if it is a homomorphic image of a free lattice such that for every $x\in L$ the set $\phi^{-1}(x)$ of preimages has a least and a greatest element. A finite lattice L is said to be *semidistributive* if it satisfies the following two implications for all $x,y,z\in L$:

$(\mathrm{SD}_\vee)$ $\quad x \vee y=x \vee z$ implies $x \vee y=x \vee (y \wedge z)$,

$(\mathrm{SD}_\wedge)$ $\quad x \wedge y=x \wedge z$ implies $x \wedge y=x \wedge (y \vee z)$.

For a finite lattice L these properties can be recognized by the arrow structure of $\mathbb{K}(L)$. It is known that every bounded lattice is semidistributive. The converse is false. Examples of nonbounded semidistributive lattices can be found in [10] or in [6]. A proof of the following characterization can be found in [6].

Lemma 3.4. *Let L be a finite lattice.*

(1) *L is semidistributive if and only if the cross-table representing* $\mathbb{K}(L)$ *contains exactly one double-arrow in each row and in each column.*

(2) *L is bounded if and only if the context $\mathbb{K}(L)$ can be represented by a square cross-table such that there are only double-arrows on the diagonal, there are only arrows up below the diagonal, and there are only arrows down above the diagonal.*

Combining Lemmas 3.2 and 3.4 we get the following theorem.

Theorem 3.5. *For all $n \in \mathbb{N}$, the lattice T_n is bounded. Therefore, T_n is semidistributive.*

This theorem was first proved by Urquhart [13]. He used his topological representation theory for the proof. As a corollary we get immediately Markowsky's result that the modular lattice M_3 is not a sublattice of T_n for $n \in \mathbb{N}$. The only canidates for sublattices are bounded lattices because the class of all bounded lattices is a pseudovariety. Conversely, we conjecture that every bounded lattice is isomorphic to a sublattice of T_n for some $n \in \mathbb{N}$. In [8], it was proven that all distributive lattices with n join-irreducibles are sublattices of T_n. Let us state the following conjecture.

Conjecture 3.6. *Let L be a finite bounded lattice with $n = |J(L)|$. Then L is isomorphic to a sublattice of T_n.*

The verification of this conjecture would yield that the class of Tamari lattices does not satisfy any equation because the subdirectly irreducible bounded lattices generate the variety of all lattices (cf. [3]).

In [4], Day proved that the class of all finite bounded lattices coincides with the class of those finite lattices which can be generated by the interval doubling construction starting from $\mathbb{2}$, the 2-element lattice. We recall the definition: given a finite lattice L, replace the interval $I \subseteq L$ by $I \times \mathbb{2}$, choose the natural order on $I \times \mathbb{2}$ and $L - I$ and define

$$x \leqslant y (x \geqslant y) \;:\Leftrightarrow\; x \leqslant y_1 (x \geqslant y_1) \text{ for } x \in L, \quad y = (y_1, i) \in I \times \mathbb{2}.$$

The resulting lattice with universe $(L - I) \dot{\cup} (I \times \mathbb{2})$ is denoted by $L[I]$. Now, the doubling of an interval can be recognized by the structure of the appropriate context. This has been worked out in a more general situation in [6] from which we deduce the following result.

Proposition 3.7. *Let L be a finite lattice and $I := [a, b] \subseteq L$ be an interval. The context $\mathbb{K}(L[I])$ is isomorphic to $\mathbb{K}(L)[I] := (J(L) \cup \{g_{[a)}\}, M(L) \cup \{m_{(b]}\}, J)$ with*

$$gJm \;:\Leftrightarrow\; \begin{cases} g \leqslant m & \text{if } g \in J(L),\ m \in M(L), \\ m \in [a) & \text{if } g = g_{[a)},\ m \in M(L), \\ g \in (b] & \text{if } g \in J(L),\ m = m_{(b]}. \end{cases}$$

In particular, we get $g'_{[a)}=[a)\cap M(L)$ and $m'_{(b]}=(b]\cap J(L)$. Thus, the interval can be determined within the context. Moreover, we can describe the arrow structure in the following way: we have $g_{[a)} \swarrow\!\!\!\nearrow m_{(b]}$, $\swarrow \cap(\{g_{[a)}\}\times M(L))=\emptyset$, and $\nearrow \cap(J(L)\times\{m_{(b]}\}=\emptyset$. Thus, in the corresponding cross-table there are no arrows down in row $g_{[a)}$, there are no arrows up in column $m_{(b]}$, except one double-arrow in the 'diagonal cell'. This applies to the context of Tamari lattices. Therefore, we get recursive lattice construction of T_{n+1} from T_n.

Theorem 3.8. *For $n\in\mathbb{N}$, define a series of n contexts by $\mathbb{K}_n^0:=\mathbb{K}(T_n)$, $\mathbb{K}_n^1:=\mathbb{K}(T_n)[I_1],\ldots,\mathbb{K}_n^n:=\mathbb{K}_n^{n-1}[I_n]$ with $I_1=T_n$, and $I_l=[[(n+1)-l,n],\langle(n+2)-l,n\rangle]$ in the lattice $\mathfrak{B}(\mathbb{K}_n^l)$ $(l=2,\ldots,n)$. Then $\mathbb{K}(T_{n+1})\cong\mathbb{K}_n^n$. In this way, T_{n+1} can be constructed from T_n by n successive doublings.*

Proof. By induction on l we can prove with the proposition above that $\mathbb{K}_n^l$ is isomorphic to $\mathbb{K}(T_{n+1})$ restricted to the join-irreducibles and meet-irreducibles in the first $(n(n-1)/2)+l$ rows and columns, respectively. We use the isomorphism that identifies the new object in step l with $[(n+1)-l, n+1]$ and the new attribute with $\langle(n+1)-l,n\rangle$. $\square$

We will illustrate this theorem in the appendix. Fig. 6 starts with T_4 and $\mathbb{K}(T_4)$. The doubling procedure can be seen step by step in Fig. 7–10, where the lattices and the appropriate contexts are given. Thus, Fig. 10b contains a picture of T_5 with 42 elements.

4. The arrow graph of Tamari lattices

The structural knowledge about the arrow relations can be concentrated in the arrow graph of a given context. This enables the study of the congruence lattice of Tamari lattices. We get the result that, for all $n\in\mathbb{N}$, T_n and $Con(T_n)$, the congruence lattice of T_n, have the same number of elements. The following definition is a modification of the original one in [5] suited to the case of semidistributive lattices. Here the term component is still used, although it consists in this paper only of one element.

Definition 4.1. Let L be a finite semidistributive lattice. The vertices of the arrow graph are all pairs (g,m) with $g\in J(L)$, $m\in M(L)$ and $g \swarrow\!\!\!\nearrow m$. We call (g,m) a *double-arrow component* and denote the set of all double-arrow components by $\swarrow\!\!\!\nearrow(\mathbb{K}(L))$. A double-arrow component (g,m) is uniquely determined by g. Thus we use the notation $\swarrow\!\!\!\nearrow(g):=(g,m)$ for $g\in J(L)$. We introduce two sorts of directed edges on $\swarrow\!\!\!\nearrow(\mathbb{K}(L))$: $(g,m)\xrightarrow{+}(h,n)$ *if* $g \nearrow n$ in $\mathbb{K}(L)$ and $g\neq h$; dually, $(g,m)\xrightarrow{-}(h,n)$ if $h \swarrow m$ in $\mathbb{K}(L)$ and $g\neq h$. The directed bigraph $(\swarrow\!\!\!\nearrow(\mathbb{K}(L)), \xrightarrow{+}, \xrightarrow{-})$ is called the *arrow-graph* of L.

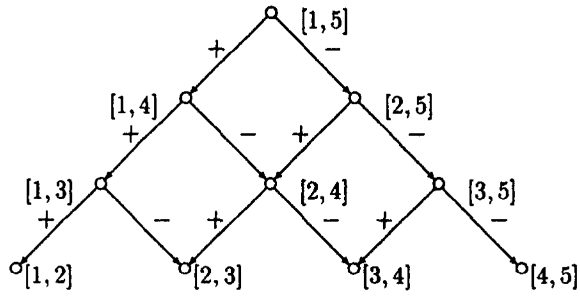


Fig. 2. The arrow graph of $\mathbb{K}(T_5)$.

We use $(g,m) \to (h,n)$ as an abbreviation for $(g,m) \xrightarrow{+} (h,n)$ or $(g,m) \xrightarrow{-} (h,n)$. For the sake of simplicity we omit transitive edges of the same sort.

It turns out that the arrow graph of T_n has a very regular structure. We put all vertices $\swarrow\!\!\nearrow([j,k])$ with $k-j=l$ on the same label $(l=1,\ldots,n-1)$ and order them in a lexicographic way. From a vertex $\swarrow\!\!\nearrow([j,k])$ with $k-j>1$ there is a plus edge in south-west direction to $\swarrow\!\!\nearrow([j,k-1])$ and a minus edge in south-east direction to $\swarrow\!\!\nearrow([j+1,k])$, both one label below. Without transitive edges there are no other edges. Fig. 2 contains the arrow graph of T_5. For notational simplicity, we denote $\swarrow\!\!\nearrow([j,k])$ by $[j,k]$. Note that from left to right the subtriangle with k elements on the bottom line corresponds to the arrow graph of T_{k+1} $(k=1,\ldots,n-1)$.

For a finite lattice L a subset S of $\swarrow\!\!\nearrow(\mathbb{K}(L))$ is called *arrow-closed* if $(g,m)\in S$ and $(g,m)\to(h,n)$ implies $(h,n)\in S$. The set of all arrow-closed subsets forms a complete lattice under set inclusion. $\swarrow\!\!\nearrow(\mathbb{K}(L))$ is *1-generated* if there exists a vertex (g,m) such that $\swarrow\!\!\nearrow(\mathbb{K}(L))=\{(h,n)\mid \text{there is a finite sequence } (g,m)\to\cdots\to(h,n)\}$. These concepts are important because of the following result due to Wille (cf. [15]).

Lemma 4.2. (1) *The lattice of all arrow-closed subsets of* $\swarrow\!\!\nearrow(\mathbb{K}(L))$ *is dually isomorphic to the congruence lattice of L.*

(2) *L is subdirectly irreducible if and only if* $(\swarrow\!\!\nearrow(\mathbb{K}(L)), \xrightarrow{+}, \xrightarrow{-})$ *is 1-generated.*

Hence, the description of the arrow graph of T_n yields the following corollary.

Corollary 4.3. *For $n\in\mathbb{N}$, lattice T_n is subdirectly irreducible.*

As an important tool we use the bijection β between $J(L)$ and the set $\swarrow\!\!\nearrow(\mathbb{K}(L))$ given by $\beta(g):=\swarrow\!\!\nearrow(g)$ for $g\in J(L)$. In a finite lattice L we have $x=\bigvee\{g\in J(L)\mid g\leqslant x\}$. Hence, x is determined by the set $J_x:=\{g\in J(L)\mid g\leqslant x\}$ which we call the *extent* of x. In the case of Tamari lattices we want to characterize those subsets in the arrow graph which correspond to lattice elements.

Proposition 4.4. *Let $S \subseteq \swarrow\!\!\nearrow(\mathbb{K}(T_n))$ be a subset of the arrow graph of T_n. We have $S = J_x$ for an extent J_x of a lattice element $x \in T_n$ if and only if S satisfies the following two conditions:*

$$(A_+) \qquad \swarrow\!\!\nearrow([j,k]) \in S,\ \swarrow\!\!\nearrow([j,k]) \xrightarrow{+} \swarrow\!\!\nearrow([i,l]) \ \Rightarrow\ \swarrow\!\!\nearrow([i,l]) \in S,$$

$$(A_\vee) \qquad \swarrow\!\!\nearrow([j,k]) \in S,\ \swarrow\!\!\nearrow([i,k+1]) \in S,\ \ j<i\leqslant k \ \Rightarrow\ \swarrow\!\!\nearrow([j,k+1]) \in S.$$

Proof. We have $\swarrow\!\!\nearrow([j,k]) \xrightarrow{+} \swarrow\!\!\nearrow([i,l])$ if and only if $j=i$ and $l<k$, but this is equivalent to $[j,k]>[i,l]$ in T_n. So S satisfies (A_+) if and only if $\beta^{-1}(S)$ is an order ideal in $(J(L),\leqslant)$. Now, let $\swarrow\!\!\nearrow([j,k])$ and $\swarrow\!\!\nearrow([i,k+1]) \in S$ with $j<i\leqslant k$. Then $[j,k] \vee [i,k+1] > [j,k+1]$ (use the vector representation) and so $\swarrow\!\!\nearrow([j,k+1]) \in S$. Thus, the conditions are necessary.

For the proof of sufficiency, first remark that $(A_\vee)$ is equivalent to the more general condition

$$(A'_\vee) \quad \swarrow\!\!\nearrow([j,k]) \in S,\ \ \swarrow\!\!\nearrow([i,l]) \in S,\ \ j<i\leqslant k<l \Rightarrow \swarrow\!\!\nearrow([j,l]) \in S.$$

By condition (A_+) we get that $\swarrow\!\!\nearrow([i,h]) \in S$ for $h=k+1,\dots,l$. Thus, $(A'_\vee)$ follows by induction on h in which $(A_\vee)$ is used as the induction step. We have to prove that every element $[j,k] \in T_n$ with $[j,k] \leqslant \bigvee \beta^{-1}(S)$ is contained in $\beta^{-1}(S)$. So, consider a join of elements $[i_s,l_s] \in \beta^{-1}(S)$ $(s=1,\dots,r)$ lexicographically ordered with

$$[j,k] < [i_1,l_1] \vee \cdots \vee [i_r,l_r].$$

Assume furthermore that none of these elements can be omitted. We get $i_1=j$ and if $r=1$ then $k<l_1$. This case is covered by condition (A_+). Moreover, if $r>1$ then $i_s<i_{s+1}\leqslant l_s$, $l_s<k$ for $s=1,\dots,r-1$, and $l_r \geqslant k$ by the algebraic description of the join in the vector representation. The case $r=2$ is covered by condition $(A'_\vee)$. Now, with the definition of the join in T_n we get

$$[i_1,l_{r-1}] < [i_1,l_1] \vee \cdots \vee [i_{r-1},l_{r-1}]$$

and proceeding with induction on r this implies $[i_1,l_{r-1}] \in \beta^{-1}(S)$. Thus, we have $[i_1,l_{r-1}] \in \beta^{-1}(S)$, $[i_r,l_r] \in \beta^{-1}(S)$ and ome more application of $(A'_\vee)$ yields $[i_r,l_r] \in \beta^{-1}(S)$. We have $[i_r,l_r]=[j,l_r]$ *and* $k \leqslant l_r$. Hence, applying condition (A_+) we get $\swarrow\!\!\nearrow([j,k]) \in S$. $\square$

The set $\beta(J_x)$ for some lattice element $x \in T_n$ can be easily visualized within the arrow graph: $\beta(J_x)$ must be closed with respect to the south-west direction. Moreover, $\swarrow\!\!\nearrow([j_1,k_1])$ is the maximal element in north-east direction if and only if there are no elements below the vertex $\swarrow\!\!\nearrow([j_1,k_1+1])$ in south-east direction.

The proposition above enables us to define a bijection between T_n and the corresponding congruence lattice. This will be done by introducing a bijection between T_n and the set of all arrow-closed subsets of $(\swarrow\!\!\!\nearrow(\mathbb{K}(T_n)), \xrightarrow{+}, \xrightarrow{-})$.

Theorem 4.5. *For T_n, there is a bijection between the lattice elements and the set of all arrow-closed subsets of the arrow graph of T_n. Thus, we get*

$$|T_n| = |\mathrm{Con}(T_n)|.$$

Proof. We define a mapping α which assigns to each set $S := \beta(J_x)$ an arrow-closed set $\alpha(\beta(J_x))$. For $k = 2, \ldots, n$ and $P \subseteq \swarrow\!\!\!\nearrow(\mathbb{K}(T_n))$ we define numbers $c_k(P) := |\{j \mid \swarrow\!\!\!\nearrow([j,k]) \in P\}|$. If $c_k(S) = 0$ then $\alpha_k(S) := \emptyset$. If $c_k(S) > 0$ then the set $\alpha_k(S)$ is defined by

$$\alpha_k(S) := \{\swarrow\!\!\!\nearrow([j,k]) \mid j = k - c_k(S), \ldots, k-1\}.$$

Finally,

$$\alpha(S) := \bigcup_{k=2}^{n} \alpha_k(S).$$

Claim. *$\alpha(S)$ is arrow-closed.*

If $\swarrow\!\!\!\nearrow([j,k]) \xrightarrow{-} \swarrow\!\!\!\nearrow([i,l])$ then $\swarrow\!\!\!\nearrow([i,l]) = \swarrow\!\!\!\nearrow([j+1,k])$. Let $\swarrow\!\!\!\nearrow([j,k]) \in \alpha(S)$. Then the definition of $\alpha(S)$ implies $\swarrow\!\!\!\nearrow([j+1,k]) \in \alpha(S)$. We remark that for every set $S = \beta(J_x)$ we have $c_{k-1}(S) \geq c_k(S) - 1$ because if $j \neq k-1$ and $\swarrow\!\!\!\nearrow([j,k]) \in S$ then (A_+) implies $\swarrow\!\!\!\nearrow([j,k-1]) \in S$. We assume $\swarrow\!\!\!\nearrow([j,k]) \xrightarrow{+} \swarrow\!\!\!\nearrow([j,k-1])$ and $\swarrow\!\!\!\nearrow([j,k]) \in \alpha(S)$. This yields $j \geq k - c_k(S)$ and with the inequality above we get $j \geq k - (c_{k-1}(S) + 1)$. This is equivalent to $j \geq (k-1) - c_{k-1}(S)$ which implies $\swarrow\!\!\!\nearrow([j,k-1]) \in \alpha(S)$.

Now, starting with an arrow-closed subset R we will assign a set $\gamma(R)$ which represents a lattice element $x \in T_n$. For $k = 2, \ldots, n$ the set $\gamma_k(R)$ is defined recursively. $\gamma_2(R)$ is given by $\gamma_2(R) := R \cap \{\swarrow\!\!\!\nearrow([1,2])\}$. For $k = 2, \ldots, n-1$, the set $\gamma_{k+1}(R)$ contains the first $c_{k+1}(R)$ elements of

$$\gamma_{k+1}(R)^+ := \{\swarrow\!\!\!\nearrow([j,k+1] \mid \swarrow\!\!\!\nearrow([j,k]) \in \gamma_k(R) \text{ or } j = k\}$$

with respect to the lexicographic order. Finally,

$$\gamma(R) := \bigcup_{k=2}^{n} \gamma_k(R).$$

Claim. *$\gamma(R) = \beta(J_x)$ for some $x \in T_n$.*

For the numbers $c_k(R)$ we get again $c_{k-1}(R) \geq c_k(R) - 1$ because R is closed with respect to $\xrightarrow{+}$. Let us assume that $\swarrow\!\!\!\nearrow([j,k]) \xrightarrow{+} \swarrow\!\!\!\nearrow([j,k-1])$ and $\swarrow\!\!\!\nearrow([j,k]) \in \gamma(R)$. We

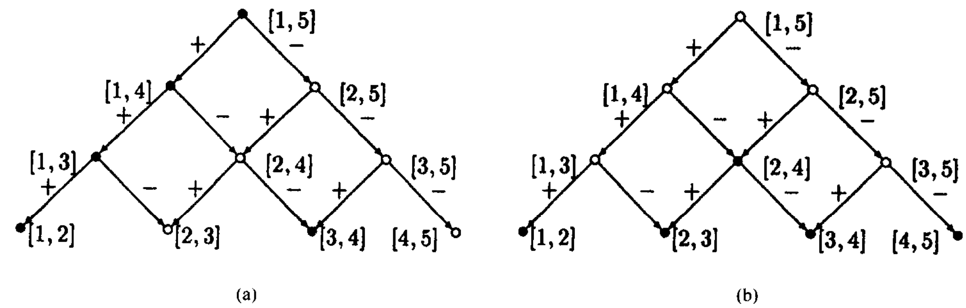


Fig. 3.(a) $S=\gamma(\alpha(S))$, (b) $\gamma(S)$.

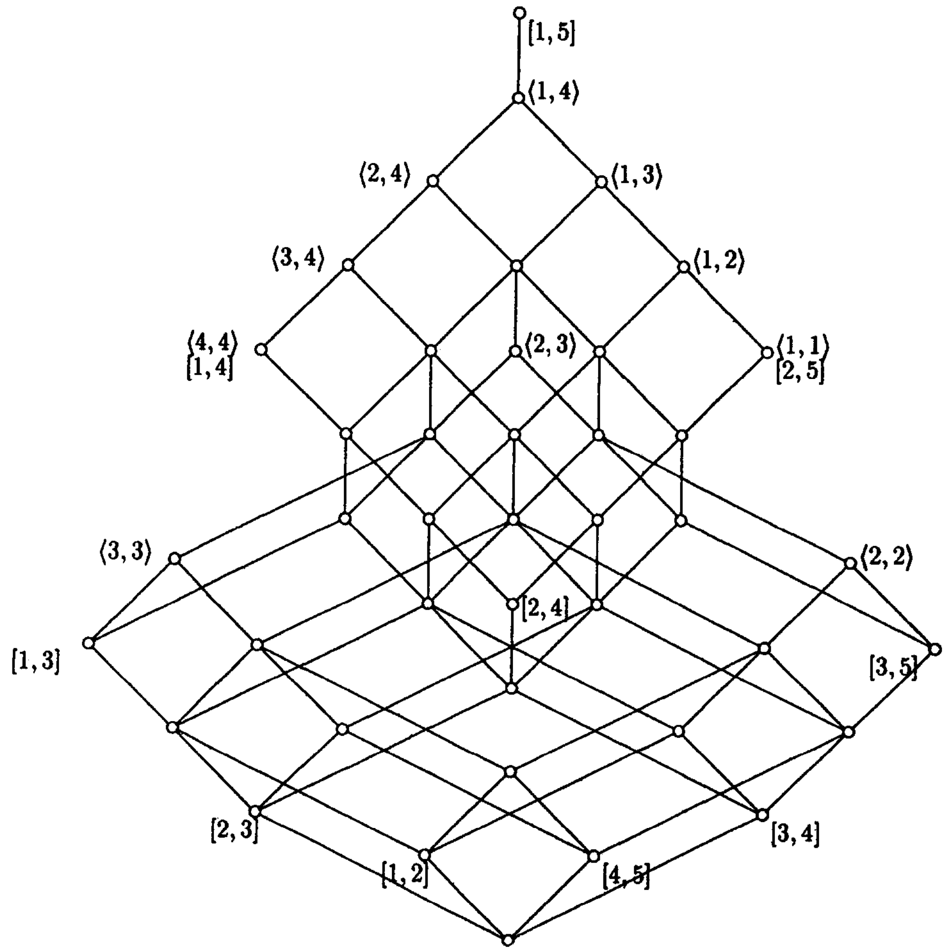


Fig. 4. The lattice of arrow-closed subsets of $\swarrow\!\!\!\nearrow(\mathbb{K}(T_5))$.

get $j \neq k-1$ and by definition of $\gamma_k(R)$ this yields $\swarrow\!\!\nearrow([j,k-1]) \in \gamma_{k-1}(R) \subseteq \gamma(R)$. This shows, that $\gamma(R)$ satisfies (A_+). Next, assume $\swarrow\!\!\nearrow([j,k]) \in \gamma(R)$, $\swarrow\!\!\nearrow([i,k+1]) \in \gamma(R)$ and $j<i \leqslant k$. $\swarrow\!\!\nearrow([j,k]) \in \gamma_k(R)$ implies $\swarrow\!\!\nearrow([j,k+1]) \in \gamma_{k+1}(R)^+$. Because of $j<i$ the element $\swarrow\!\!\nearrow([j,k+1])$ precedes $\swarrow\!\!\nearrow([i,k+1])$ in the lexicographic order. Hence, $\swarrow\!\!\nearrow([i,k+1]) \in \gamma_{k+1}(R)$ together with $c_{k-1}(R) \geqslant c_k(R)-1$ implies $\swarrow\!\!\nearrow([j,k+1]) \in \gamma_{k+1}(R) \subseteq \gamma(R)$ and so condition $(A_\vee)$ is also satisfied.

For a subset $P \subseteq \swarrow\!\!\nearrow(\mathbb{K}(T_n))$ both mappings are uniquely determined by the numbers $c_k(P)$ for $k=2,\ldots,n$. Moreover, for every $k \in \{2,\ldots,n\}$ we get $c_k(P)=c_k(\alpha(P))=c_k(\gamma(P))$ which proves $\alpha^{-1}=\gamma$. We get the desired bijection if we recall in mind the bijection between lattice elements and their extent representation, the bijection β between $J(L)$ and the corresponding double-arrow components, and the dual isomorphism between arrow-closed subsets and congruence relations. $\square$

Fig. 3 contains an example demonstrating the construction above. Fig. 4 represents the lattice of arrow-closed subsets of $\swarrow\!\!\nearrow(\mathbb{K}(T_5))$.

5. Catalan numbers

Tamari lattices describe an ordering of all binary bracketings. The nth Catalan number C_n is given by $C_n := |T_n|$, the number of all binary bracketings on $n+1$ symbols (cf. [2]). In the last section we described a bijection between the lattice elements of T_n and the arrow-closed subsets of the arrow graph $(\swarrow\!\!\nearrow(\mathbb{K}(T_n)), \xrightarrow{+}, \xrightarrow{-})$. If we denote by $\geqslant_n$ the reflexive transitive hull of the relation $\rightarrow$ then $(\swarrow\!\!\nearrow(\mathbb{K}(T_n)), \geqslant_n)$ is an ordered set, which we call the *triangle order* Tr_n. In this interpretation, arrow-closed subsets correspond to order ideals. Hence, we get the following corollary.

Corollary 5.1. *C_n is the number of order ideals in the triangle order Tr_n.*

Analysing the doubling procedure in detail, we prove that the number of coverings in T_n is given by $((n-1)/2)C_n$. For this purpose, we recall some facts concerning formal concept analysis. Let $I=[a,b]$ be an interval of L and let $\mathbb{K}(L)$ be the underlying context. I is a sublattice of L with context $\mathbb{K}_I = (J(L) \cap (b], M(L) \cap [a), \leqslant)$. To get a reduced context we take only those join-irreducibles in $J(L) \cap (b]$ which contain a double-arrow in $M(L) \cap [a)$ and dually. For two contexts (G_1, M_1, I_1) and (G_2, M_2, I_2) we define the sum $(G_1, M_1, I_1)+(G_2, M_2, I_2)$by

$$(G_1, M_1, I_1)+(G_2, M_2, I_2) := (G_1 \dot\cup G_2, M_1 \dot\cup M_2, I_1 \dot\cup I_2 \dot\cup (G_1 \times M_2) \dot\cup (G_2 \times M_1)).$$

If L_1 and L_2 are finite lattices then the reduced context corresponding to the direct product of these lattices is given by $\mathbb{K}(L_1 \times L_2) \cong \mathbb{K}(L_1) + \mathbb{K}(L_2)$.

We need the following lemma describing the intervals connected with the doublings.

Lemma 5.2. *Consider the n doublings starting from T_n yielding T_{n+1} as described in Theorem 3.8. Then $I_1 \cong T_n$ and, for $k=2,\ldots,n$, we get $I_k \cong T_{(n+1)-k} \times T_{k-1}$, where T_1 is the one-element lattice.*

Proof. By Theorem 3.8, we have $I_1 = T_n$. Throughout the proof we identify $\mathbb{K}_n^l$ with $\mathbb{K}(T_{n+1})$ restricted to the join-irreducibles and meet-irreducibles in the first $(n(n-1)/2)+l$ rows and columns, respectively. Assume $l \in \{2,\ldots,n\}$. The interval I_l is given by $I_l = [[(n+1)-l,n], \ \langle (n+2)-l,n\rangle] \subseteq \mathfrak{B}(\mathbb{K}_n^l)$. We have $\langle (n+2)-l,n\rangle' \cap J(T_n) = J(T_n)$. Using the remark above, the reduced context to $\mathbb{K}_l \subseteq \mathbb{K}(T_n)$ is determined by $[(n+1)-l,n]' \cap M(T_n)$ and the join-irreducibles connected with them via the double-arrow relation. We define

$$M := \{m \in M(T_n) \mid [(n+1)-l,n+1] \leqslant m \text{ in } \mathbb{K}_n^l\},$$

$$G := \{g \in J(T_n) \mid (\exists m \in M) g \swarrow\!\!\!\nearrow m \text{ in } \mathbb{K}_n^l\}.$$

We have to show that $(G,M,\leqslant)$ is isomorphic to the direct sum $\mathbb{K}(T_{(n+1)-l}) + \mathbb{K}(T_{l-1})$. We first observe that

$$[(n+1)-l,n]' \cap M(T_{(n+1)-l}) = M(T_{(n+1)-l}) = \{\langle p,q\rangle \in M(T_n) \mid (n+1)-l \nleqslant q\}$$

because $\langle p,q\rangle \in M(T_{(n+1)-l})$ implies $p,q \in \{1,\ldots,n-l\}$. Thus, $(n+1)-l \nleqslant q$ and so Proposition 2.4 implies the result. With $G_1 := J(T_{(n+1)-l}) \subseteq G$ and $M_1 := M(T_{(n+1)-l}) \subseteq M$ we obtain $(G_1, M_1, \leqslant) \cong \mathbb{K}(T_{(n+1)-l})$. Next, we define

$$M_2 := [(n+1)-l,n+1]' \cap (M(T_n) \setminus M(T_{(n+1)-l}))$$

$$= \{\langle p,q\rangle \in M(T_n) \mid (n+1)-l \leqslant p,q\},$$

$$G_2 := \{[j,k] \in J(T_n) \mid (\exists m \in M_2)[j,k] \swarrow\!\!\!\nearrow m \text{ in } \mathbb{K}_n^l\}$$

$$= \{[j,k] \in J(T_n) \mid (n+1)-l < j < k \leqslant n\}.$$

The pair of mappings $\iota_G : G_2 \to J(T_{l-1})$ given by $\iota_G([j,k]) := [j-(n+1-l), k-(n+1-l)]$ and $\iota_M : M_2 \to M(T_{l-1})$ given by $\iota_M(\langle p,q\rangle) := \langle p-(n+1-l), q-(n+1-l)\rangle$ provides a context isomorphism which proves $(G_2, M_2, \leqslant) \cong \mathbb{K}(T_{l-1})$. We are done if $G_1 \times M_2 \subseteq \leqslant$ and $G_2 \times M_1 \subseteq \leqslant$. Assume $[j,k] \in G_2$ and $\langle p,q\rangle \in M_1$. Then $p,q \in \{1,\ldots,n-l\}$ but $(n+1)-l<j$. Hence, $j \nleqslant q$ and so Proposition 2.4 yields the result. Now, if $[j,k] \in G_1$ and $\langle p,q\rangle \in M_2$ we get $j<(n+1)-l<p$, so again $[j,k] \leqslant \langle p,q\rangle$ by the definition of the (non)incidence in Proposition 2.4. Altogether, we obtain $(G,M,\leqslant) \cong (G_1,M_1,\leqslant)+(G_2,M_2,\leqslant) \cong \mathbb{K}(T_{(n+1)-l})) + \mathbb{K}(T_{l-1})$. $\square$

Now, we can state the following theorem.

Theorem 5.3. *The number of coverings in T_n is given by $(n-1)/2\,|T_n| = ((n-1)/2)C_n$.*

Proof. We will count the number of new coverings in each step of the doubling construction. The proof will be done by induction on n. First, we remark that the number of elements in $T_k \times T_l$ is given by $|T_k| \cdot |T_l|$ while the number of coverings can be computed using induction as

$$|T_k| \cdot \frac{l-1}{2} \cdot |T_l| + |T_l| \cdot \frac{k-1}{2} \cdot |T_k| = |T_l| \cdot |T_k| \cdot \frac{1}{2}(l+k-2)$$

$$= \frac{1}{2}(l+k-2) C_k \cdot C_l.$$

In each doubling step we get as many coverings in the new direction as elements in the choosen interval while the coverings within the interval are copied. Thus, we also have to add the number of coverings in this interval. We start with T_n which has C_n elements and $((n-1)/2) \cdot C_n$ coverings. So, after the first step we have $((n-1)/2) \cdot C_n + C_n + ((n-1)/2) \cdot C_n = n \cdot C_n$ coverings. By Lemma 5.2 the doubled intervals are isomorphic to $T_{n+1-k} \times T_{k-1}$ for $k = 2, \ldots, n$. With the observation above the number of new coverings in step k turns out to be

$$|T_{(n+1)-k}| \cdot |T_{k-1}| + |T_{(n+1)-k}| \cdot |T_{k-1}| \cdot \frac{(n+1)-k+(k-1)-2}{2}$$

$$= \frac{n}{2} \cdot C_{(n+1)-k} C_{k-1}.$$

So, using the recursive formula for the Catalan numbers (see [2]) we get

$$n \cdot C_n + \sum_{k=2}^{n} \frac{n}{2} C_{n+1-k} \cdot C_{k-1} = \sum_{k=1}^{n+1} \frac{n}{2} C_{n+1-k} \cdot C_{k-1}$$

$$= \frac{n}{2} \cdot C_{n+1}$$

as the total number of coverings in T_{n+1}. □

We close this section with the proof that also the congruence lattice of T_n has $((n-1)/2) \cdot |Con(T_n)| = ((n-1)/2) \cdot C_n$ coverings. So, besides the Boolean lattices, Tamari lattices provide a second series of lattices where the number of vertices and the number of edges in the Hasse diagram of the lattice and the appropriate congruence lattice coincide. The proof will be done by counting the number of k-generated ideals in the triangle order Tr_n. They correspond to arrow-closed subsets with k lower neighbours. Thus, the number of coverings in the lattice of arrow-closed subsets is given by multiplying each ideal with the number of its generators. The dual isomorphism between the lattice of arrow-closed subsets and the congruence lattice yields the desired result.

We will get $((n-1)/2)\cdot C_n$ elements if we can show that the number of k-generated and the number of $(n-1-k)$-generated ideals coincide ($k=0,\ldots,n-1$). This will be done in the following proposition. The exact value of k-generated ideals in Tr_n is given by the so-called *Narayana numbers* (cf. [9, 12])[1].

Proposition 5.4. *The number of k-generated ideals in the triangle order Tr_n is given by the Narayana number* $u(n,k+1):=\frac{1}{n}\binom{n}{k+1}\binom{n}{k}$.

Proof. We consider the part P of the integer lattice with coordinates in $\{0,\ldots,n\}$ which lies on or below the diagonal $\{(k,\ n-k)\,|\,k=0,\ldots,n\}$. We identify the triangle Tr_n with all lattice points with coordinates in $\{1,\ldots,n-1\}$. Now, consider all lattice paths from $(0,n)$ to $(n,0)$ which do not exceed the diagonal, i.e., which are totally contained in P. For $k\in\{0,\ldots,n-1\}$, we identify a k-generated ideal with generators on the lattice points (x_1,y_1), $(x_2,y_2),\ldots,(x_k,y_k)(x_1<x_2<\cdots<x_k)$ with the path

$$(0,n)\to(0,y_1)\to(x_1,y_1)\to(x_1,y_2)\to\cdots\to(x_k,y_k)\to(x_k,0)\to(n,0).$$

This clearly gives a bijection between the k-generated order ideals and all paths with $k+1$ segments going down. The 0-generated ideal is identified with the path $(0,n)\to(0,0)\to(n,0)$. We encode each path by a sequence of $+$ and $-$ signs: for each step down we write a plus sign while each step to the right is encoded by a minus sign. When speaking of partial sums we identify $+$ with $+1$ and $-$ with -1. The path does not exceed the diagonal if and only if partial sums in the corresponding sequence are positive. A maximal string of signs of the same sort standing on successive places will be called a *run*. Thus, the number of k-generated ideals equals the number of such sequences containing $k+1$ runs of plus signs and $k+1$ runs of minus signs. We count them using the following trick: add a new minus sign at place $2n+1$. Now, we have the following theorem of Bogart: For every sequence of n plus signs and $n+1$ minus signs, there is one and only one way to cyclically shift the sequence so that all partial sums up to $2n$ summands are positive and a minus sign stands at place $2n+1$.

Now, we are ready to count the number of sequences with $k+1$ runs. We have to partition the n plus signs into $k+1$ parts. There are $n-1$ spaces between them so this can be done in $\binom{n-1}{k}$ different ways. Similarly, the $n+1$ minus signs can be partitioned in $\binom{n}{k}$ many ways. Each sequence associated with a "correct" path has to start with a plus sign. By cyclic permutation, $k+1$ of these sequences are equivalent. By Bogart's theorem, there is exactly one sequence under these which has positive partial sums up to $2n$ summands. This one has to end with a minus sign, the one which we added for simplifying the enumeration. So, the number of correct paths with $k+1$ runs

[1] I have to thank K. Bogart for the main ideas of this proof and fruitful discussions during his visit in Darmstadt.

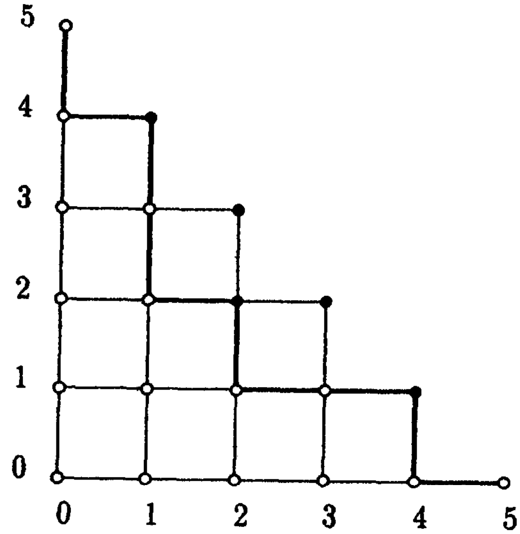


Fig. 5. The path corresponding to $\alpha(S)$ in Fig. 3.

is given by

$$u(n,k+1)=\frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}=\frac{1}{n}\binom{n}{k+1}\binom{n}{k}. \qquad \square$$

Fig. 5 contains as an example the path corresponding to the set $\alpha(S)$ in Fig. 3. Obviously,we have

$$\begin{aligned} u(n,k+1)&=\frac{1}{n}\binom{n}{k+1}\binom{n}{k}\\ &=\frac{1}{n}\binom{n}{n-(k+1)}\binom{n}{n-k}\\ &=u(n,n-k). \end{aligned}$$

So, this gives us as a corollary.

Corollary 5.5. *For the Tamari lattice T_n we get $|T_n|=|Con(T_n)|=C_n$ and the number of coverings in T_n and $Con(T_n)$ equal both $((n-1)/2)C_n$.*

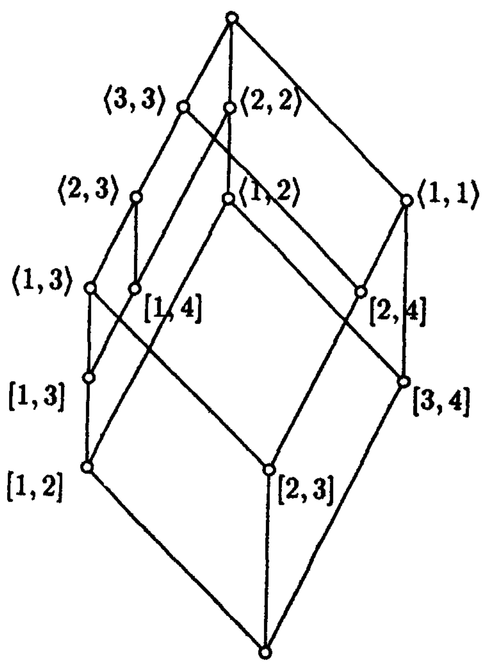
A third series of lattices where the number of elements is given by the Catalan numbers, are the lattices of noncrossing partitions (cf. [11]). Tamari lattices are semi-distributive, their congruence lattices are distributive while the lattice of noncrossing partitions as a sublattice of the partition lattice fails to have these properties. These lattices are graded, atomistic and co-atomistic. Recently, Ganter found a recursive construction of the appropriate contexts.

Appendix A: The doubling construction for T_5

We now illustrate Theorem 3.8 through a series of Figs. 6–10.

	⟨1,1⟩	⟨2,2⟩	⟨1,2⟩	⟨3,3⟩	⟨2,3⟩	⟨1,3⟩
[1,2]	⤢	×	×	×	×	×
[2,3]	×	⤢	↙	×	×	×
[1,3]	↗	×	⤢	×	×	×
[3,4]	×	×	×	⤢	↙	↙
[2,4]	×	↗		×	⤢	↙
[1,4]	↗	×	↗	×	×	⤢

(a)

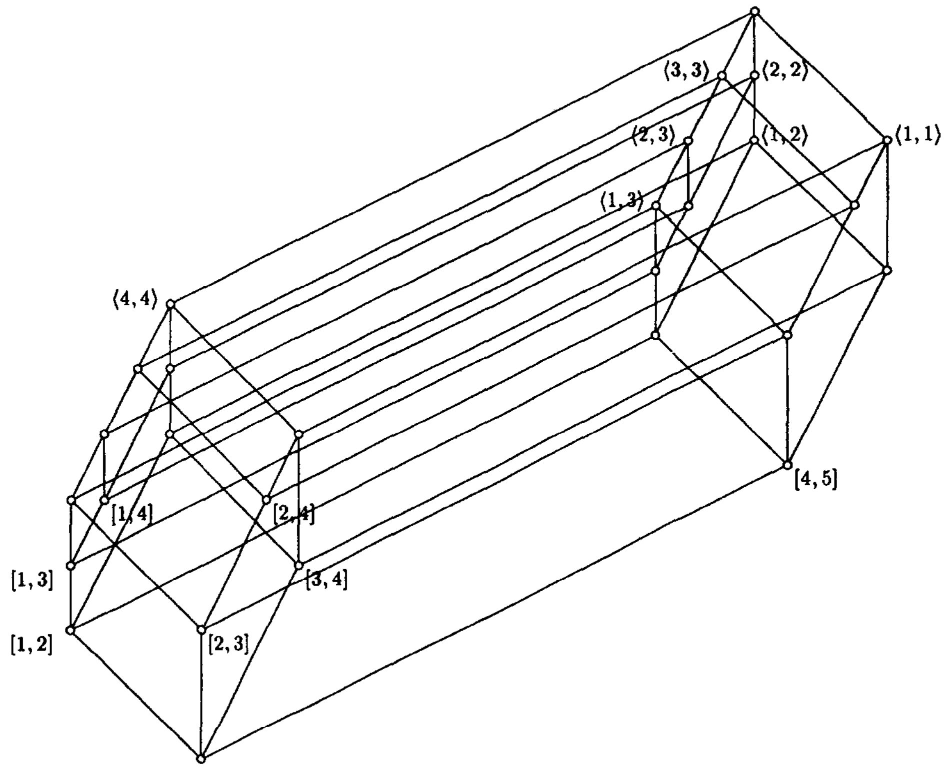


(b)

Fig. 6(a). $\mathbb{K}(T_4)$, and (b). T_4.

	⟨1, 1⟩	⟨2, 2⟩	⟨1, 2⟩	⟨3, 3⟩	⟨2, 3⟩	⟨1, 3⟩	⟨4, 4⟩
[1, 2]	⤢	×	×	×	×	×	×
[2, 3]	×	⤢	↙	×	×	×	×
[1, 3]	↗	×	⤢	×	×	×	×
[3, 4]	×	×	×	⤢	↙	↙	×
[2, 4]	×	↗		×	⤢	↙	×
[1, 4]	↗	×	↗	×	×	⤢	×
[4, 5]	×	×	×	×	×	×	⤢

(a)

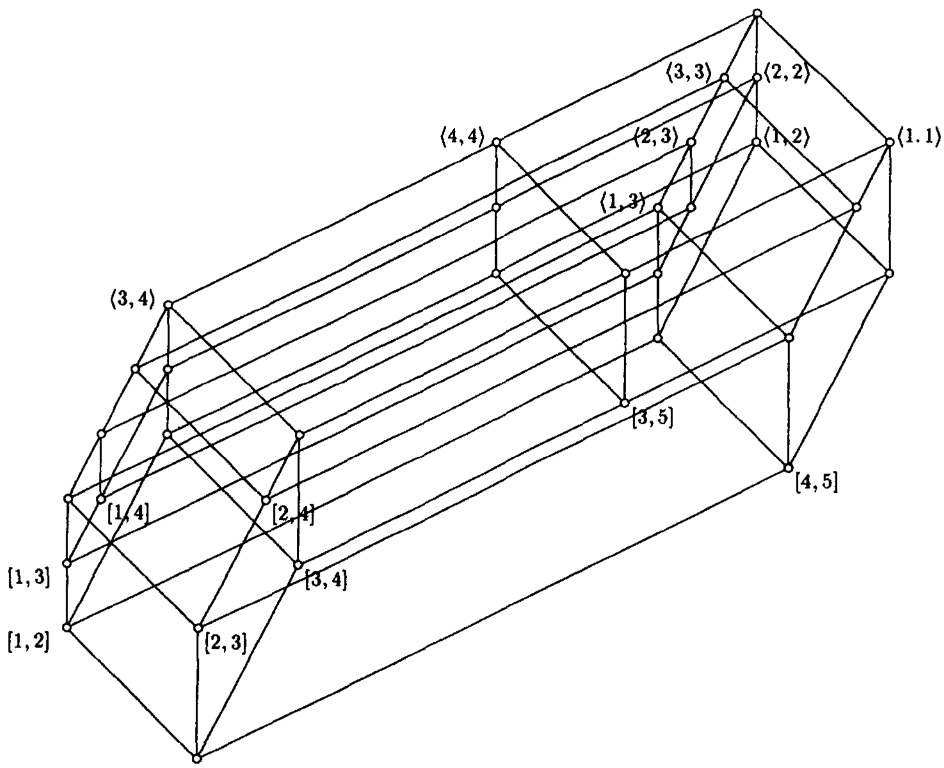


(b)

Fig. 7(a). $\mathbb{K}(T_4[T_4])$, (b). $T_4[T_4]$.

	⟨1,1⟩	⟨2,2⟩	⟨1,2⟩	⟨3,3⟩	⟨2,3⟩	⟨1,3⟩	⟨4,4⟩	⟨3,4⟩
[1,2]	⤢	×	×	×	×	×	×	×
[2,3]	×	⤢	↙	×	×	×	×	×
[1,3]	↗	×	⤢	×	×	×	×	×
[3,4]	×	×	×	⤢	↙	↙	×	×
[2,4]	×	↗		×	⤢	↙	×	×
[1,4]	↗	×	↗	×	×	⤢	×	×
[4,5]	×	×	×	×	×	×	⤢	↙
[3,5]	×	×	×	↗			×	⤢

(a)

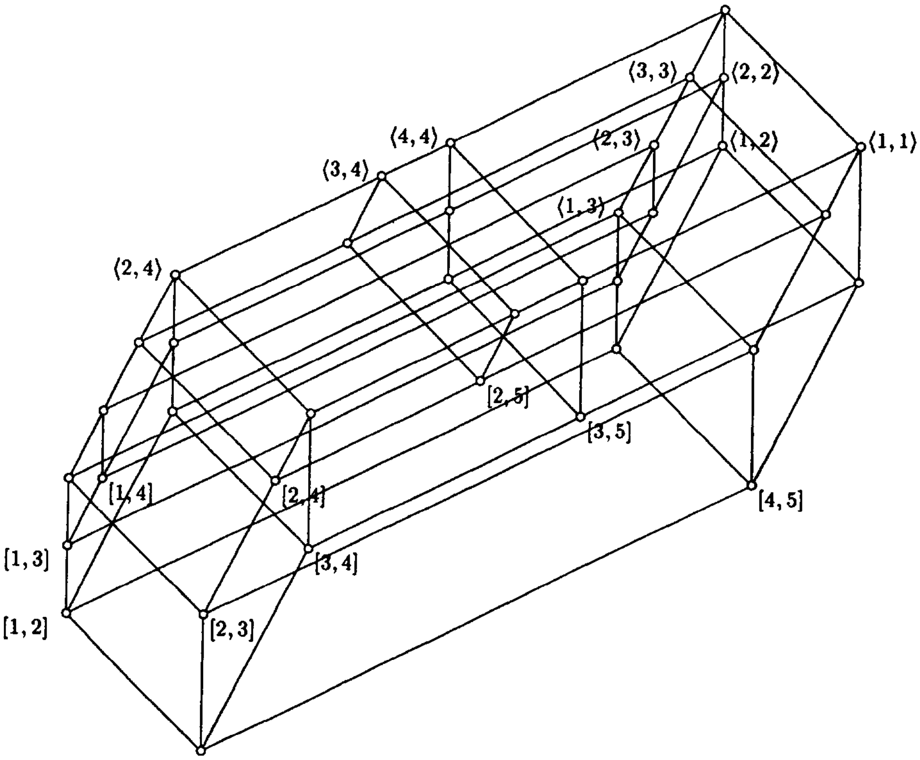


(b)

Fig. 8(a). $\mathbb{K}(T_4[T_4][([3,4]$, (b). $T_4[T_4][[[3,4],\langle 4,4\rangle]]$.

	⟨1,1⟩	⟨2,2⟩	⟨1,2⟩	⟨3,3⟩	⟨2,3⟩	⟨1,3⟩	⟨4,4⟩	⟨3,4⟩	⟨2,4⟩
[1,2]	⤢	×	×	×	×	×	×	×	×
[2,3]	×	⤢	↙	×	×	×	×	×	×
[1,3]	↗	×	⤢	×	×	×	×	×	×
[3,4]	×	×	×	⤢	↙	↙	×	×	×
[2,4]	×	↗		×	⤢	↙	×	×	×
[1,4]	↗	×	↗	×	×	⤢	×	×	×
[4,5]	×	×	×	×	×	×	⤢	↙	↙
[3,5]	×	×	×	↗			×	⤢	↙
[2,5]	×	↗		×	↗		×	×	⤢

(a)

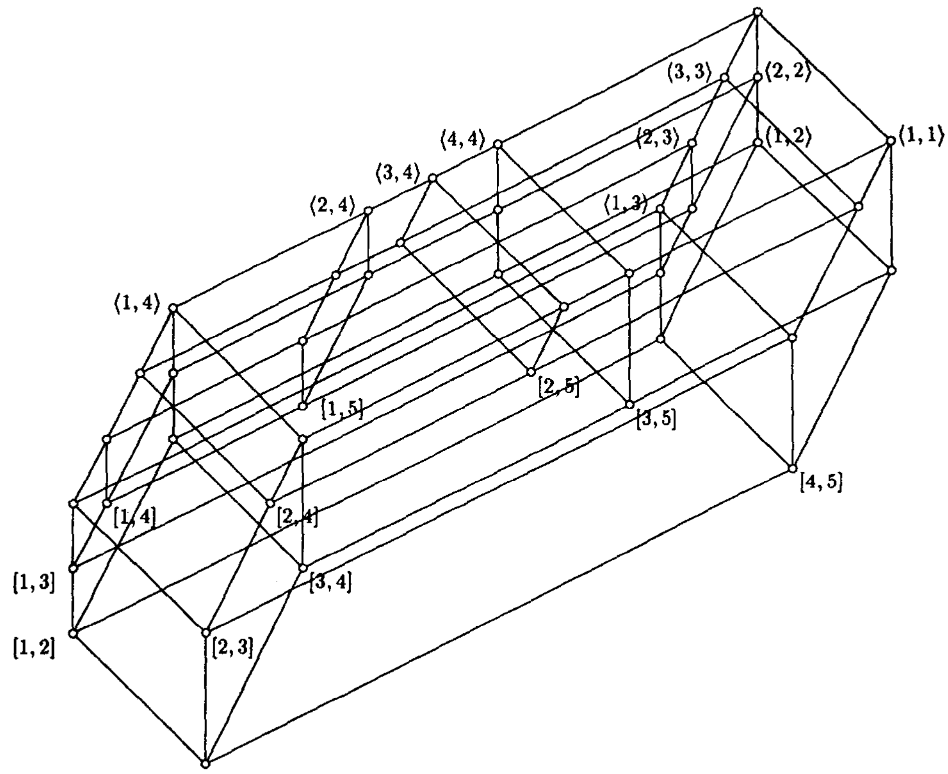


(b)

Fig. 9(a). $\mathbb{K}(T_4[T_4][([3,4],\langle 4,4\rangle)][([2,4],\langle 4,4\rangle)])$, (b). $T_4[T_4][[[3,4],\langle 4,4\rangle]][[[2,4],\langle 4,4\rangle]]$.

	⟨1,1⟩	⟨2,2⟩	⟨1,2⟩	⟨3,3⟩	⟨2,3⟩	⟨1,3⟩	⟨4,4⟩	⟨3,4⟩	⟨2,4⟩	⟨1,4⟩
[1,2]	⤢	×	×	×	×	×	×	×	×	×
[2,3]	×	⤢	$\swarrow$	×	×	×	×	×	×	×
[1,3]	$\nearrow$	×	⤢	×	×	×	×	×	×	×
[3,4]	×	×	×	⤢	$\swarrow$	$\swarrow$	×	×	×	×
[2,4]	×	$\nearrow$		×	⤢	$\swarrow$	×	×	×	×
[1,4]	$\nearrow$	×	$\nearrow$	×	×	⤢	×	×	×	×
[4,5]	×	×	×	×	×	×	⤢	$\swarrow$	$\swarrow$	$\swarrow$
[3,5]	×	×	×	$\nearrow$			×	⤢	$\swarrow$	$\swarrow$
[2,5]	×	$\nearrow$		×	$\nearrow$		×	×	⤢	$\swarrow$
[1,5]	$\nearrow$	×	$\nearrow$	×	×	$\nearrow$	×	×	×	⤢

(a)



(b)

Fig. 10(a). $\mathbb{K}(T_5)$, (b). T_5.

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