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The axioms of constructive geometry

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Abstract

Elementary geometry can be axiomatized constructively by taking as primitive the concepts of the apartness of a point from a line and the convergence of two lines, instead of incidence and parallelism as in the classical axiomatizations. I first give the axioms of a general plane geometry of apartness and convergence. Constructive projective geometry is obtained by adding the principle that any two distinct lines converge, and affine geometry by adding a parallel line construction, etc. Constructive axiomatization allows solutions to geometric problems to be effected as computer programs. I present a formalization of the axiomatization in type theory. This formalization works directly as a computer implementation of geometry.

1. The general form of a geometric problem

The general form of a geometric problem is the following: For given data x of type A, to find y such that condition $C(x, y)$ is fulfilled. The sought y is of a type that may depend on x, to be denoted $B(x)$. If y is found according to effective rules of construction, solutions to geometric problems can be realized as computer programs. Theorems appear as special cases of problems in which the sought object y is either absent or not intrinsically interesting. In classical geometry, instead, existence of y of type $B(x)$ can be proved indirectly, and it need not be the case that a method for actually finding y is produced.

I will present the axioms of constructive projective, affine and orthogonal plane geometries. Each will be an extension of a common base I call *apartness geometry*.

The theory presented here has been implemented in higher-level type theory. The correctness of the proofs of theorems as well as solutions of problems of constructive geometry can be easily checked through formalization in type theory. It also gives as a byproduct an algorithm for executing the task set in the specification of a geometric problem.

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2. Choice of basic concepts

The principle that will guide us in the choice of basic concepts is: Put the 'finitely precise' in the basic concepts, and let the 'ideally precise' be realized by construction postulates. Instead of point equality, point apartness is used, and the same for lines. Instead of incidence of a point with a line, *apartness* of a point from a line is used. It was the lack of this last concept that prevented earlier attempts, beginning with Heyting in the twenties, at axiomatizing constructive geometry in a fully satisfactory way. Further, instead of parallelism, *convergence* of two lines is used as a basic concept.

We move from apartness geometry (or incidence geometry according to classical terminology) to projective geometry by requiring that any two distinct lines converge. In affine geometry, the concept used is convergence together with a rule of construction for parallel lines. In orthogonal geometry, the basic concept to be introduced is, for lack of a better word, the *unorthogonality* of two lines. I shall formulate the axiomatization in such a way that the three geometries are obtained by adding different principles to apartness geometry, without changing anything in the common basic axioms.

The old concepts of equality, incidence, parallelism and orthogonality are defined as negations of the constructively basic concepts. Thus, parallelism, for example, means that convergence is impossible. But from impossibility of parallelism, we cannot constructively infer convergence. In analogy to the geometry of the real plane, we could say that convergence requires a positive bound, whereas denial of parallelism allows two lines to be indefinitely close to being parallel. No bound need exist for how difficult it could be to establish their convergence despite impossibility of parallelism. Similarly, impossibility of equality of two points need not give a positive bound for their distance. In logical terms, the difference between the classical and constructive axiomatizations is that the latter does not use the rule of double negation, from $\sim\sim A$ to infer A. This principle, or, equivalently, the law of excluded middle $A \vee \sim A$, leads to nonconstructive existence proofs, so we cannot maintain the distinction between the computable and the noncomputable anymore. But we require that its addition to the constructive axioms brings us back the classical ones. If the law of double negation is added, the implementation of Section 11 turns into an implementation of classical geometry. We still have a proof-checking algorithm, but the computability of solutions of geometric problems is lost.

3. The axioms and rules of apartness geometry

Points will be denoted by $a, b, c, d, \ldots$, and lines by $l, m, n, r, \ldots$. To express the assertion that a is a point, we write a: Point, and similarly l: Line for lines. The basic relations DiPt, DiLn, Con and Apt are read as follows, where $\triangleright$ indicates the translation of a formal expression into English

$\mathrm{DiPt}(a,b) \triangleright a$ and b are distinct points,

$\mathrm{DiLn}(l,m) \triangleright l$ and m are distinct lines,

$\mathrm{Con}(l, m) \rhd l$ and m are convergent lines,

$\mathrm{Apt}(a, l) \rhd$ point a is apart from line l.

An alternative for $\mathrm{DiPt}(a, b)$ is to translate it as points a and b are apart, and the same for $\mathrm{DiLn}(l, m)$.

Next we define

$$\mathrm{EqPt}(a, b) = \sim \mathrm{DiPt}(a, b),$$

$$\mathrm{EqLn}(l, m) = \sim \mathrm{DiLn}(l, m),$$

$$\mathrm{Par}(l, m) = \sim \mathrm{Con}(l, m),$$

$$\mathrm{Inc}(a, l) = \sim \mathrm{Apt}(a, l).$$

These translate as follows:

$\mathrm{EqPt}(a, b) \rhd a$ and b are coincident points,

$\mathrm{EqLn}(l, m) \rhd l$ and m are coincident lines,

$\mathrm{Par}(l, m) \rhd l$ and m are parallel lines,

$\mathrm{Inc}(a, l) \rhd$ point a is incident with line l.

Instead of coincident we shall often say equal, but it should be kept in mind that two geometric objects can then be equal without being identical. We shall see for example that lines constructed in different ways can be equal in the sense of coincidence.

Translation from formal to informal language (i.e., *sugaring* in computer science terminology) admits of degrees: In the expressions $\mathrm{EqPt}(a, b)$ and $\mathrm{EqLn}(l, m)$, for example, the propositional functions EqPt and EqLn are applied to objects of the type of points and lines, respectively. In expressions such as $\mathrm{Eq}(a, b)$ and $\mathrm{Eq}(l, m)$, the functional structure is ambiguous; the type information has to be read from the convention concerning symbols for objects. The common mathematical notation, $a = b$ and $l = m$, is still less explicit, and finally we arrive at such typical expressions of informal mathematical language as: a and b are equal, l and m are equal, and so on (see [9] for the sugaring process). In this paper I shall write out the formal expressions, possibly at the expense of readability.

It will be useful to introduce the symbols $\cdot$ and $|$, to be used as abbreviations, according to the following pattern:

$$\mathrm{DiPt}(a \cdot b, c) = \mathrm{DiPt}(a, c) \,\&\, \mathrm{DiPt}(b, c),$$

$$\mathrm{DiPt}(a \,|\, b, c) = \mathrm{DiPt}(a, c) \vee \mathrm{DiPt}(b, c)$$

and similarly for the second argument of DiPt, and the rest of the relations. We call $a \cdot b$ the *term conjunction* and $a \,|\, b$ the *term disjunction* of a and b.

The axiomatization contains rules of inference, rules of construction and axioms. The rules of logical inference are basically those of constructive logic in a natural

deduction formulation. But we move from the Gentzen-style natural deduction toward type-theoretical reasoning by using the typing notation $a : A$ for objects belonging to sets. The type-theoretical proof objects are not written out, however, until the description of the implementation in Section 11. For example, the rule of $\exists$-introduction has as one premise $a : A$ and as another $\vdash B(a)$, with bounded quantification over A in the conclusion. Note also the distinction between the proposition $B(a)$ and the assertion (or judgment) $\vdash B(a)$:

$$\frac{a : A \qquad \vdash B(a)}{\vdash (\exists x : A) B(x)}\ \exists\text{-intr}$$

This gives a practical system of reasoning that can be almost routinely put in type-theoretic form by adding a proof object whenever there is an assertion sign $\vdash$, and by taking into account that later objects and propositions may depend on these proof objects.

Leaving aside the general rules of logical inference, the axiomatization consists of rules of construction (where the classical axiomatizations would use existential axioms) and the constructive axioms proper.

Rules of construction

In apartness geometry, there will be a rule for constructing a line from two distinct points, and a rule for constructing a point from two convergent lines. These rules are also written with their premises above a line and the conclusion below:

$$\frac{a : \text{Point} \quad b : \text{Point} \quad \vdash \text{DiPt}(a, b)}{\text{ln}(a, b) : \text{Line}} \qquad \frac{l : \text{Line} \quad m : \text{Line} \quad \vdash \text{Con}(l, m)}{\text{pt}(l, m) : \text{Point}}$$

We have the translations:

$\text{ln}(a, b) \rhd$ the connecting line of points a and b,

$\text{pt}(l, m) \rhd$ the intersection point of lines l and m.

I. Apartness axioms for distinct points, distinct lines, and convergent lines

I.a Irreflexivity

$\vdash \sim \text{DiPt}(a, a)$,

$\vdash \sim \text{DiLn}(l, l)$,

$\vdash \sim \text{Con}(l, l)$.

I.b. Apartness

$\vdash \text{DiPt}(a, b) \rightarrow \text{DiPt}(a, c) \vee \text{DiPt}(b, c)$,

$\vdash \text{DiLn}(l, m) \rightarrow \text{DiLn}(l, n) \vee \text{DiLn}(m, n)$,

$\vdash \text{Con}(l, m) \rightarrow \text{Con}(l, n) \vee \text{Con}(m, n)$.

II. Axioms for connecting lines and intersection points

$$\vdash \mathrm{DiPt}(a, b) \to\ \sim \mathrm{Apt}(a, \mathrm{ln}(a, b)),$$

$$\vdash \mathrm{DiPt}(a, b) \to\ \sim \mathrm{Apt}(b, \mathrm{ln}(a, b)),$$

$$\vdash \mathrm{Con}(l, m) \to\ \sim \mathrm{Apt}(\mathrm{pt}(l, m), l),$$

$$\vdash \mathrm{Con}(l, m) \to\ \sim \mathrm{Apt}(\mathrm{pt}(l, m), m).$$

III. Constructive uniqueness axiom for lines and points

$$\vdash \mathrm{DiPt}(a, b) \,\&\, \mathrm{DiLn}(l, m) \to \mathrm{Apt}(a \mid b, l \mid m).$$

IV. Compatibility of equality with apartness and convergence

$$\vdash \mathrm{Apt}(a, l) \to \mathrm{DiPt}(a, b) \vee \mathrm{Apt}(b, l),$$

$$\vdash \mathrm{Apt}(a, l) \to \mathrm{DiLn}(l, m) \vee \mathrm{Apt}(a, m),$$

$$\vdash \mathrm{Con}(l, m) \to \mathrm{DiLn}(m, n) \vee \mathrm{Con}(l, n).$$

We shall not always write out the symbol $\vdash$ in the following.

The idea behind the apartness principle in I.b is that if two points a and b are distinct, $\mathrm{DiPt}(a, b)$, they are some finite distance apart. Therefore, if it requires ideal precision to decide whether a and c are distinct, it must be the case that b and c are some finite distance apart, so that $\mathrm{DiPt}(b, c)$ can be inferred.

Relations satisfying axioms of type I.a and b are symmetric: Assume $\mathrm{DiPt}(a, b)$. Substituting a for c in I.b gives $\mathrm{DiPt}(a, b) \to \mathrm{DiPt}(a, a) \vee \mathrm{DiPt}(b, a)$, so $\mathrm{DiPt}(a, a) \vee \mathrm{DiPt}(b, a)$ follows. Since $\sim \mathrm{DiPt}(a, a)$ by I.a, $\mathrm{DiPt}(b, a)$.

The negation of a relation satisfying the principles I.a and b, say EqPt, has the following properties. By I.a, $\mathrm{EqPt}(a, a)$, and by the contraposition of I.b, $\mathrm{EqPt}(a, b) \,\&\, \mathrm{EqPt}(a, c) \to \mathrm{EqPt}(b, c)$. By the contraposition of the symmetry of DiPt, EqPt is symmetric; thus, negations of apartness relations are equivalence relations.

The axiom III includes as a consequence the uniqueness of constructed lines and points:

Theorem 3.1 (Uniqueness of constructed lines)

$$\mathrm{DiPt}(a, b) \,\&\, \mathrm{Inc}(a \cdot b, l) \to \mathrm{EqLn}(l, \mathrm{ln}(a, b)).$$

Proof. Assume $\mathrm{DiPt}(a, b) \,\&\, \mathrm{Inc}(a \cdot b, l)$. Since $\mathrm{Inc}(a, \mathrm{ln}(a, b))$ and $\mathrm{Inc}(b, \mathrm{ln}(a, b))$ by II, we have $\mathrm{DiPt}(a, b) \,\&\, \mathrm{Inc}(a \cdot b, l \cdot \mathrm{ln}(a, b))$. From III we get $\mathrm{DiPt}(a, b) \,\&\, \mathrm{Inc}(a \cdot b, l \cdot m) \to \mathrm{EqLn}(l, m)$. The substitution $[\mathrm{ln}(a, b)/m]$ now gives

$$\mathrm{DiPt}(a, b) \,\&\, \mathrm{Inc}(a \cdot b, l \cdot \mathrm{ln}(a, b)) \to \mathrm{EqLn}(l, \mathrm{ln}(a, b)),$$

so that $\mathrm{EqLn}(l, \mathrm{ln}(a, b))$ follows. $\square$

From the compatibility axioms IV it follows easily that equal objects can be substituted in the apartness and incidence relations. If $\mathrm{Apt}(a, l)$ and if $\mathrm{EqPt}(a, b)$, it follows from the first axiom in IV that $\mathrm{Apt}(b, l)$. By the contraposition of the same axiom, substitution of equal points holds also for incidence. From the second axiom in IV we get $\mathrm{Apt}(a, m)$ from $\mathrm{Apt}(a, l)$ and $\mathrm{EqLn}(l, m)$. Further, from the third axiom in IV we get $\mathrm{Con}(l, n)$ from $\mathrm{Con}(l, m)$ and $\mathrm{EqLn}(m, n)$, so that substitution of equal lines holds also for the convergence relation. By the contrapositive of the third axiom in IV, substitution holds for parallels. Finally, the compatibility axiom for convergence gives by the substitution $[l/n]$ $\mathrm{Con}(l, m) \to \mathrm{DiLn}(m, l) \vee \mathrm{Con}(l, l)$, but $\sim \mathrm{Con}(l, l)$ by I, so that line convergence implies line apartness, and line equality implies parallelism:

Theorem 3.2. $\mathrm{Con}(l, m) \to \mathrm{DiLn}(l, m)$.

Theorem 3.3 (Uniqueness of constructed points)

$$\mathrm{Con}(l, m) \,\&\, \mathrm{Inc}(a, l \cdot m) \to \mathrm{EqPt}(a, \mathrm{pt}(l, m)).$$

Proof. Similar to Theorem 3.1. □

4. Symmetry in apartness and incidence

Lemma 4.1. *Assume* $\mathrm{DiPt}(a, b)$, $\mathrm{Con}(l, m)$. *Then*

(i) $\mathrm{DiLn}(l, \ln(a, b)) \leftrightarrow \mathrm{Apt}(a \mid b, l)$.

(ii) $\mathrm{DiPt}(a, \mathrm{pt}(l, m)) \leftrightarrow \mathrm{Apt}(a, l \mid m)$.

Proof. (i) If $\mathrm{DiLn}(l, \ln(a, b))$ axiom III implies $\mathrm{Apt}(a \mid b, l \mid \ln(a, b))$, but $\sim \mathrm{Apt}(a, \ln(a, b))$ and $\sim \mathrm{Apt}(b, \ln(a, b))$ by II, so $\mathrm{Apt}(a \mid b, l)$.

If $\mathrm{Apt}(a, l)$, by IV, $\mathrm{DiLn}(l, \ln(a, b)) \vee \mathrm{Apt}(a, \ln(a, b))$. By II, $\sim \mathrm{Apt}(a, \ln(a, b))$, so $\mathrm{DiLn}(l, \ln(a, b))$. If $\mathrm{Apt}(b, l)$, similarly $\mathrm{DiLn}(l, \ln(a, b))$.

(ii) is proved similarly. □

Theorem 4.2 (Symmetry of Apt). *Assume* $\mathrm{DiPt}(a, b)$, $\mathrm{DiPt}(c, d)$. *Then*

$$\mathrm{Apt}(a \mid b, \ln(c, d)) \to \mathrm{Apt}(c \mid d, \ln(a, b)).$$

Proof. Assume $\mathrm{Apt}(a \mid b, \ln(c, d))$. By Lemma 4.1 we get $\mathrm{DiLn}(\ln(c, d), \ln(a, b))$. By symmetry of DiLn, $\mathrm{DiLn}(\ln(a, b), \ln(c, d))$. By Lemma 4.1 again, $\mathrm{Apt}(c \mid d, \ln(a, b))$. □

Lemma 4.3. *Assume* $\mathrm{DiPt}(a, b)$. *Then*

(i) $\mathrm{Apt}(c, \ln(a, b)) \to \mathrm{DiPt}(c, a) \,\&\, \mathrm{DiPt}(c, b)$.

(ii) $\mathrm{Apt}(c, \ln(a, b)) \to \mathrm{DiLn}(\ln(a, b), \ln(c, a)) \,\&\, \mathrm{DiLn}(\ln(a, b), \ln(c, b))$.

Proof. (i) Assume $\mathrm{Apt}(c, \mathrm{ln}(a, b))$. $\mathrm{Apt}(c, \mathrm{ln}(a, b)) \to \mathrm{DiPt}(c, a) \vee \mathrm{Apt}(a, \mathrm{ln}(a, b))$ by IV, but $\sim \mathrm{Apt}(a, \mathrm{ln}(a, b))$ by II, so $\mathrm{DiPt}(c, a)$. Similarly $\mathrm{DiPt}(c, b)$.

(ii) By IV, $\mathrm{Apt}(c, \mathrm{ln}(a, b)) \to \mathrm{DiLn}(\mathrm{ln}(a, b), \mathrm{ln}(c, a)) \vee \mathrm{Apt}(c, \mathrm{ln}(c, a))$, but $\sim \mathrm{Apt}(c, \mathrm{ln}(c, a))$ by II, so $\mathrm{DiLn}(\mathrm{ln}(a, b), \mathrm{ln}(c, a))$. Similarly $\mathrm{DiLn}(\mathrm{ln}(a, b), \mathrm{ln}(c, b))$. □

Theorem 4.4 (Triangle axioms). *Assume* $\mathrm{DiPt}(a, b)$.

(i) $\mathrm{Apt}(c, \mathrm{ln}(a, b)) \to \mathrm{Apt}(a, \mathrm{ln}(c, b))$.
(ii) $\mathrm{Apt}(c, \mathrm{ln}(a, b)) \to \mathrm{Apt}(b, \mathrm{ln}(a, c))$.
(iii) $\mathrm{Apt}(c, \mathrm{ln}(a, b)) \to \mathrm{Apt}(c, \mathrm{ln}(b, a))$.

Proof. (i) Assume $\mathrm{Apt}(c, \mathrm{ln}(a, b))$. By Lemma 4.3, $\mathrm{DiPt}(c, b)$, so that $\mathrm{ln}(c, b)$: Line. By same lemma $\mathrm{DiLn}(\mathrm{ln}(a, b), \mathrm{ln}(c, b))$. Since $\mathrm{DiPt}(a, b)$, the apartness axiom III gives $\mathrm{Apt}(a \mid b, \mathrm{ln}(a, b) \mid \mathrm{ln}(c, b))$, of which only $\mathrm{Apt}(a, \mathrm{ln}(c, b))$ remains.

(ii) Similar to (i).

(iii) Assume $\mathrm{Apt}(c, \mathrm{ln}(a, b))$. By (i) $\mathrm{Apt}(a, \mathrm{ln}(c, b))$, by (ii) $\mathrm{Apt}(b, \mathrm{ln}(c, a))$, by (i) $\mathrm{Apt}(c, \mathrm{ln}(b, a))$. □

The symmetry results of Theorem 4.4 are similar to what have been called triangle axioms in earlier literature.

For incidence, we get the corresponding results.

Lemma 4.5. *Assume* $\mathrm{DiPt}(a, b)$, $\mathrm{Con}(l, m)$. *Then*

(i) $\mathrm{EqLn}(l, \mathrm{ln}(a, b)) \leftrightarrow \mathrm{Inc}(a \cdot b, l)$.
(ii) $\mathrm{EqPt}(a, \mathrm{pt}(l, m)) \leftrightarrow \mathrm{Inc}(a, l \cdot m)$.

Proof. From Lemma 4.1. □

Lemma 4.5 is another rendering of the uniqueness of constructed lines and points, as in Theorems 3.1 and 3.3.

Theorem 4.6. *Assume* $\mathrm{DiPt}(a, b)$, $\mathrm{DiPt}(c, d)$. *Then*

$$\mathrm{Inc}(a \cdot b, \mathrm{ln}(c, d)) \to \mathrm{Inc}(c \cdot d, \mathrm{ln}(a, b)).$$

Proof. Contraposition of Theorem 4.2. □

Corollary 4.7. *Assume* $\mathrm{DiPt}(a, b)$, $\mathrm{DiPt}(a, c)$, $\mathrm{DiPt}(b, c)$.

(i) $\mathrm{Inc}(c, \mathrm{ln}(a, b)) \to \mathrm{Inc}(a, \mathrm{ln}(c, b))$.
(ii) $\mathrm{Inc}(c, \mathrm{ln}(a, b)) \to \mathrm{Inc}(b, \mathrm{ln}(a, c))$.
(iii) $\mathrm{Inc}(c, \mathrm{ln}(a, b)) \to \mathrm{Inc}(c, \mathrm{ln}(b, a))$.

In terms of the intersection point constructor pt, the remaining duals for Theorems 4.2–4.7 are the following.

Theorem 4.8 (Symmetry of apartness). *Assume* Con(l, m), Con(n, r). *Then*

$$\mathrm{Apt}(\mathrm{pt}(l, m), n \mid r) \rightarrow \mathrm{Apt}(\mathrm{pt}(n, r), l \mid m).$$

Lemma 4.9. *Assume* Con(l, m). *Then*

$$\mathrm{Apt}(\mathrm{pt}(l, m), n) \rightarrow \mathrm{DiLn}(l, n) \,\&\, \mathrm{DiLn}(m, n).$$

Corollary 4.10. *Assume* Con(l, m), Con(n, m), Con(l, n). *Then*

(i) $\mathrm{Apt}(\mathrm{pt}(l, m), n) \rightarrow \mathrm{Apt}(\mathrm{pt}(n, m), l)$.
(ii) $\mathrm{Apt}(\mathrm{pt}(l, m), n) \rightarrow \mathrm{Apt}(\mathrm{pt}(l, n), m)$.
(iii) $\mathrm{Apt}(\mathrm{pt}(l, m), n) \rightarrow \mathrm{Apt}(\mathrm{pt}(m, l), n)$.

In contrast to the triangle axioms 4.4, these principles do not seem to have been singled out in previous literature.

Theorem 4.11. *Assume* Con(l, m), Con(n, r). *Then*

$$\mathrm{Inc}(\mathrm{pt}(l, m), n \cdot r) \rightarrow \mathrm{Inc}(\mathrm{pt}(n, r), l \cdot m).$$

Corollary 4.12. *Assume* Con(l, m), Con(n, m), Con(l, n). *Then*

(i) $\mathrm{Inc}(\mathrm{pt}(l, m), n) \rightarrow \mathrm{Inc}(\mathrm{pt}(n, m), l)$.
(ii) $\mathrm{Inc}(\mathrm{pt}(l, m), n) \rightarrow \mathrm{Inc}(\mathrm{pt}(l, n), m)$.
(iii) $\mathrm{Inc}(\mathrm{pt}(l, m), n) \rightarrow \mathrm{Inc}(\mathrm{pt}(m, l), n)$.

5. Construction and coincidence

If EqPt(a, b), points a and b are equal in the sense that 'they occupy the same place', to use a phrase of Euclid, though they need not be identical by way of construction. Similarly, EqLn(l, m) means the lines l and m are coincident, though they need not be constructed in the same way. The first results along this line are

Theorem 5.1. *Assume* DiPt(a, b). *Then*

$$\mathrm{EqLn}(\ln(a, b), \ln(b, a)).$$

Proof. By axiom II, $\mathrm{Inc}(b \cdot a, \ln(a, b))$, and by Lemma 4.5(i) $\mathrm{Inc}(b \cdot a, \ln(a, b)) \rightarrow \mathrm{EqLn}(\ln(a, b), \ln(b, a))$.

Theorem 5.2. *Assume* Con(l, m). *Then*

$$\mathrm{EqPt}(\mathrm{pt}(l, m), \mathrm{pt}(m, l)).$$

Proof. Dual to the previous. □

Theorem 5.3. *Assume* DiPt(a, b), DiPt(a, c) *and* Con(ln(a, b), ln(a, c)). *Then*

$$\mathrm{EqPt}(\mathrm{pt}(\mathrm{ln}(a, b), \mathrm{ln}(a, c)), a).$$

Proof. Assume DiPt(pt(ln(a, b), ln(a, c)), a). By axiom III,

$$\mathrm{Apt}(a \mid \mathrm{pt}(\mathrm{ln}(a, b), \mathrm{ln}(a, c)), \mathrm{ln}(a, b) \mid \mathrm{ln}(a, c)).$$

But

$$\sim \mathrm{Apt}(a, \mathrm{ln}(a, b)),$$

$$\sim \mathrm{Apt}(a, \mathrm{ln}(a, c)),$$

$$\sim \mathrm{Apt}(\mathrm{pt}(\mathrm{ln}(a, b), \mathrm{ln}(a, c)), \mathrm{ln}(a, b)),$$

$$\sim \mathrm{Apt}(\mathrm{pt}(\mathrm{ln}(a, b), \mathrm{ln}(a, c)), \mathrm{ln}(a, c)).$$

Therefore EqPt(pt(ln(a, b), ln(a, c)), a). □

Theorem 5.4. *Assume* Con(l, m), Con(l, n) *and* DiPt(pt(l, m), pt(l, n)). *Then*

$$\mathrm{EqLn}(\mathrm{ln}(\mathrm{pt}(l, m), \mathrm{pt}(l, n)), l).$$

Proof. Dual to the previous. □

Theorem 5.5. *Assume* DiPt(a, b) *and* EqPt(b, c). *Then*

$$\mathrm{DiPt}(a, c) \quad \textit{and} \quad \mathrm{EqLn}(\mathrm{ln}(a, b), \mathrm{ln}(a, c)).$$

Proof. By I.b, DiPt$(a, b) \to$ DiPt$(a, c) \vee$ DiPt(b, c), but since EqPt(b, c), DiPt(a, c) follows. Assume now DiLn(ln(a, b), ln(a, c)). By III, Apt$(a \mid c$, ln$(a, b) \mid$ ln(a, c)). Axiom II eliminates all but Apt$(c$, ln(a, b)). Since by compatibility axiom IV Apt$(c$, ln$(a, b)) \,\&\,$ EqPt$(b, c) \to$ Apt$(b$, ln(a, b)), DiLn(ln(a, b), ln(a, c)) is impossible. □

Theorem 5.6. *Assume* Con(l, m) *and* EqLn(m, n). *Then*

$$\mathrm{Con}(l, n) \quad \textit{and} \quad \mathrm{EqPt}(\mathrm{pt}(l, m), \mathrm{pt}(l, n)).$$

Proof. Analogous to the previous. □

6. Projective geometry

The axioms of constructive plane projective geometry are obtained by adding to the previous rules of construction and axioms I–IV the following:

P1. Projective axiom

$$\vdash \mathrm{DiLn}(l, m) \rightarrow \mathrm{Con}(l, m).$$

The contraposition $\mathrm{Par}(l, m) \rightarrow \mathrm{EqLn}(l, m)$ gives the classical statement: All parallels to a given line are to be equated. Since we have $\mathrm{Con}(l, m) \rightarrow \mathrm{DiLn}(l, m)$ in the general apartness geometry, the concept of line apartness becomes redundant in projective geometry. Usually it is organized the other way around, by a rule that allows the construction of an intersection point for any two distinct lines. Some comments are in order on this matter.

The principle of duality is usually stated as follows. If in a theorem you interchange points and lines, including the interchange of intersection points and connecting lines, you get another, dual theorem. But the statement is incomplete, for the relations *distinct* and *equal* have no independent standing. When formalized, it is seen that the duality involves the further interchange of the propositional functions DiPt and DiLn. The duality we use, and which was implicit already in Section 4, is one where DiPt and Con are interchanged. It is no more complicated than interchanging DiPt and DiLn and it has two advantages: Nothing need be changed when we proceed from apartness geometry to the projective, affine or other geometries. Each can instead be seen as a different specialization of the general geometry of apartness. Secondly, the duality extends right from apartness geometry through projective to affine and other geometries.

A way to circumvent the need to change the old construction postulate for intersection points is to introduce *points at infinity*. With such points, the postulate applies under the condition $\mathrm{DiLn}(l, m)$ not only in projective but also in affine geometry. The old duality is maintained, but the undecidability of parallelism reappears in the question whether an intersection point is a point at infinity or an ordinary point. A constructive geometry with ideal objects can be developed but we shall not pursue the matter further here.

As an example from plane projective geometry, the problem of finding a *projectivity* is solved in Section 9.

7. Affine geometry

Since parallelism is an 'ideally precise' notion, it has to be effected by a construction postulate according to our basic principle in Section 2. Instead of the projective axiom P1, we shall add to the previous rules of construction and the axioms I–IV a postulate

to such an effect (a 'parallel ruler'), as well as the constructive axioms of parallel lines:

Rule of construction for parallel lines

$$\frac{l\colon \mathrm{Line} \quad a\colon \mathrm{Point}}{\mathrm{par}(l, a)\colon \mathrm{Line}}$$

We have the translation

$\mathrm{par}(l, a) \rhd$ the parallel to line l through point a.

The axioms are

A1. Axioms for constructed parallels

$$\vdash \sim \mathrm{Con}(\mathrm{par}(l, a), l),$$

$$\vdash \sim \mathrm{Apt}(a, \mathrm{par}(l, a)).$$

The apartness properties of Con, and consequently the equivalence properties of its negation, are contained already in the axioms of apartness geometry. As a last axiom, we have

A2. Constructive uniqueness axiom for parallels

$$\vdash \mathrm{DiLn}(l, m) \rightarrow \mathrm{Apt}(a, l\,|\,m) \vee \mathrm{Con}(l, m).$$

Some care is needed in the handling of free parameters. For example, assuming the antecedent $\mathrm{DiLn}(l, m)$ of axiom A2, we get $\mathrm{Apt}(a, l\,|\,m) \vee \mathrm{Con}(l, m)$. If $\mathrm{Par}(l, m)$, $\mathrm{Apt}(a, l\,|\,m)$ follows. In the other direction, $\mathrm{Apt}(a, l\,|\,m)$ implies $\mathrm{Par}(l, m)$. For if $\mathrm{Con}(l, m)$, construct $\mathrm{pt}(l, m)$ and you have $\sim \mathrm{Apt}(\mathrm{pt}(l, m), l \cdot m)$; therefore $\sim \mathrm{Con}(l, m)$. Still, the consequent of axiom A2 is not equivalent to $\mathrm{Par}(l, m) \vee \mathrm{Con}(l, m)$ (which would be an instance of the law of excluded middle) since an implication of the form $\forall x(A(x) \vee B) \rightarrow \forall x A(x) \vee B$ fails constructively. The constructive motivation for axiom A2 follows the 'finiteness' principle of Section 2.

With axiom A2, the classical expression of the uniqueness of the parallel construction can be derived:

Theorem 7.1. $\mathrm{Inc}(a, l) \,\&\, \mathrm{Par}(l, m) \rightarrow \mathrm{EqLn}(l, \mathrm{par}(m\ a))$.

Proof. Assume $\mathrm{Inc}(a, l)$, $\mathrm{Par}(l, m)$. Since $\mathrm{Par}(m, \mathrm{par}(m, a))$, we get $\mathrm{Par}(l, \mathrm{par}(m, a))$. Since $\mathrm{Inc}(a, \mathrm{par}(m, a))$, we get $\mathrm{Inc}(a, l \cdot \mathrm{par}(m, a))$. The contraposition of the uniqueness axiom is, with $\mathrm{par}(m, a)$ substituted for m, $\mathrm{Inc}(a, l \cdot \mathrm{par}(m, a)) \,\&\, \mathrm{Par}(l, \mathrm{par}(m, a)) \rightarrow \mathrm{EqLn}(l, \mathrm{par}(m, a))$, which proves the theorem. □

By using the parallel construction, we can now derive the irreflexivity of line convergence, or axiom I.a for Con, from the remaining axioms of affine geometry:

Theorem 7.2 (Irreflexivity of line convergence). $\sim \mathrm{Con}(l, l)$.

Proof. From the apartness axiom I.b for Con it follows easily that $\mathrm{Con}(l, m) \,\&\, \mathrm{Par}(m, n) \rightarrow \mathrm{Con}(l, n)$. The substitution $[l/m,\ \mathrm{par}(l, a)/n]$ gives $\mathrm{Con}(l, l) \,\&\, \mathrm{Par}(l, \mathrm{par}(l, a)) \rightarrow \mathrm{Con}(l, \mathrm{par}(l, a))$. Assume now $\mathrm{Con}(l, l)$. Since axiom A1 gives $\mathrm{Par}(l, \mathrm{par}(l, a))$ we conclude $\mathrm{Con}(l, \mathrm{par}(l, a))$, therefore $\sim \mathrm{Con}(l, l)$. □

Note that in addition to the line l, the proof requires a point a to be given.

Lemma 7.3. *Assume* $\mathrm{Inc}(a, m)$, $\mathrm{Inc}(a, n)$ *and* $\mathrm{Par}(m, n)$. *Then* $\mathrm{EqLn}(m, n)$.

Proof. By Theorem 7.1, $\mathrm{EqLn}(\mathrm{par}(n, a), m)$ and $\mathrm{EqLn}(\mathrm{par}(n, a), n)$, so by transitivity $\mathrm{EqLn}(m, n)$. □

Theorem 7.4. *Assume* $\mathrm{Apt}(a, l)$, $\mathrm{Inc}(a, m)$, $\mathrm{Inc}(a, n)$ *and* $\mathrm{Par}(m, l)$ *and* $\mathrm{Par}(n, l)$. *Then* $\mathrm{EqLn}(m, n)$.

Proof. By symmetry and transitivity of Par, $\mathrm{Par}(m, l)$ and $\mathrm{Par}(n, l)$ give $\mathrm{Par}(m, n)$. By the lemma, $\mathrm{EqLn}(m, n)$. □

In Theorem 7.4 we recognize the usual form of the uniqueness of the axiom of parallels. Its derivation uses the classical principle of the transitivity of parallels, that is, the contraposition of the apartness axiom I.b for convergent lines. Note that the latter axiom is constructively stronger than the classical principle, but classically equivalent to it. In the classical formulation, it is also customary to make the redundant hypothesis $\mathrm{Apt}(a, l)$, which is not needed in the proof. The reason is that one traditionally does not consider a line parallel to itself. But failure of $\mathrm{Apt}(a, l)$ does not make our parallel line construction redundant: Even if the lines l and $\mathrm{par}(l, a)$ were coincident, they would not be the same line.

Examples of the solution of problems in affine geometry are given in Section 9.

8. Orthogonality

We shall now add to affine geometry a constructive axiomatization of the concept of orthogonality. As the basic relation, a positive notion of unorthogonality can be chosen. Orthogonality is defined as its negation. The correct constructive axioms for

unorthogonality are

O1. Compatibility of convergence and unorthogonality

$$\vdash \mathrm{Con}(l, m) \vee \mathrm{Unort}(l, m).$$

O2. Apartness axiom for the conjunction of convergence and unorthogonality

$$\vdash \mathrm{Con}(l, m) \,\&\, \mathrm{Unort}(l, m) \rightarrow (\mathrm{Con}(l, n) \,\&\, \mathrm{Unort}(l, n)) \vee (\mathrm{Con}(m, n) \,\&\, \mathrm{Unort}(m, n)).$$

Definition of orthogonality

$$\mathrm{Ort}(l, m) = \sim \mathrm{Unort}(l, m).$$

Orthogonality, like parallelism, is an 'ideal notion', and calls for a rule of construction:

Construction rule for orthogonal lines

$$\frac{l : \mathrm{Line} \quad a : \mathrm{Point}}{\mathrm{ort}(l, a) : \mathrm{Line}}$$

We have the reading

$$\mathrm{ort}(l, a) \rhd \text{the orthogonal to line } l \text{ through point } a.$$

O3. Axioms for the orthogonal construction

$$\vdash \sim \mathrm{Unort}(\mathrm{ort}(l, a), l),$$

$$\vdash \sim \mathrm{Apt}(a, \mathrm{ort}(l, a)).$$

O4. Constructive uniqueness axiom for orthogonals

$$\vdash \mathrm{DiLn}(l, m) \rightarrow \mathrm{Apt}(a, l \mid m) \vee \mathrm{Unort}(l \mid m, n).$$

Theorem 8.1 (Uniqueness of orthogonality)

$$\mathrm{Inc}(a, l) \,\&\, \mathrm{Ort}(l, m) \rightarrow \mathrm{EqLn}(l, \mathrm{ort}(m, a)).$$

Proof. Assume $\mathrm{Inc}(a, l)$, $\mathrm{Ort}(l, m)$. By axiom O3, $\mathrm{Inc}(a, \mathrm{ort}(m, a))$ and $\mathrm{Ort}(m, \mathrm{ort}(m, a))$. The contraposition of axiom O4 is $\mathrm{Inc}(a, l \cdot m) \,\&\, \mathrm{Ort}(l \cdot m, n) \rightarrow \mathrm{EqLn}(l, m)$. The substitution $[\mathrm{ort}(m, a)/m, m/n]$ gives

$$\mathrm{Inc}(a, l \cdot \mathrm{ort}(m, a)) \,\&\, \mathrm{Ort}(l \cdot \mathrm{ort}(m, a), m) \rightarrow \mathrm{EqLn}(l, \mathrm{ort}(m, a))$$

which proves the result. $\square$

Theorem 8.2. $\mathrm{Unort}(l, l)$.

Proof. Substituting l for m in O1 gives $\mathrm{Con}(l, l) \vee \mathrm{Unort}(l, l)$, so by irreflexivity of convergence, Theorem 7.2, we conclude $\mathrm{Unort}(l, l)$. □

Theorem 8.3. $\mathrm{Unort}(l, m) \rightarrow \mathrm{Con}(l, n) \vee \mathrm{Unort}(m, n)$.

Proof. Assume $\mathrm{Unort}(l, m)$. By O1, $\mathrm{Con}(m, n) \vee \mathrm{Unort}(m, n)$. If $\mathrm{Unort}(m, n)$, the conclusion follows. Assume therefore $\mathrm{Con}(m, n)$. By apartness, $\mathrm{Con}(l, m) \vee \mathrm{Con}(l, n)$. If $\mathrm{Con}(l, n)$, the conclusion follows. Assume therefore $\mathrm{Con}(l, m)$. Since $\mathrm{Unort}(l, m)$, axiom O2 gives

$$(\mathrm{Con}(l, n) \,\&\, \mathrm{Unort}(l, n)) \vee (\mathrm{Con}(m, n) \,\&\, \mathrm{Unort}(m, n)).$$

The left side implies $\mathrm{Con}(l, n)$, the right side implies $\mathrm{Unort}(m, n)$, so that the conclusion follows. □

Corollary 8.4. $\mathrm{Unort}(l, m) \rightarrow \mathrm{DiLn}(l, n) \vee \mathrm{Unort}(m, n)$.

Corollary 8.5. $\mathrm{Unort}(l, m) \rightarrow \mathrm{Unort}(m, l)$.

Proof. By substitution of l for n in the theorem. □

Theorem 8.6. $\mathrm{Con}(l, m) \rightarrow \mathrm{Unort}(l, n) \vee \mathrm{Unort}(m, n)$.

Proof. Assume $\mathrm{Con}(l, m)$. By Theorem 3.2, $\mathrm{DiLn}(l, m)$, so by axiom O4 $\mathrm{Apt}(a, l \mid m) \vee \mathrm{Unort}(l \mid m, n)$. Substituting $\mathrm{pt}(l, m)$ for a leaves only $\mathrm{Unort}(l \mid m, n)$. □

It is possible to replace axiom O1 by Theorems 8.2 and 8.3. This is seen by substituting l for m in the latter. We get

$$\mathrm{Unort}(l, l) \rightarrow \mathrm{Con}(l, n) \vee \mathrm{Unort}(l, n)$$

which with $\mathrm{Unort}(l, l)$ gives O1. If we replace O1 by Theorem 8.3 only we can have self-orthogonal lines, as in Minkowski geometry.

Axiom O2 is a principle about the conjunction of convergence and unorthogonality. This relation, defined as $\mathrm{Obl}(l, m) = \mathrm{Con}(l, m) \,\&\, \mathrm{Unort}(l, m)$, can be termed *obliqueness*. (An oblique line is, after *Oxford Advanced Learner's Dictionary*, 'not horizontal or vertical'. We relativize this by taking one of the lines to be 'horizontal'.) Assuming $\mathrm{Obl}(l, l)$, the definition gives $\mathrm{Con}(l, l)$, so that $\sim\mathrm{Obl}(l, l)$. Since we also have axiom O2, obliqueness is an apartness relation. (It is possible to give all the axioms of constructive orthogonal geometry in terms of Obl instead of Unort.)

From the above, a natural way of axiomatizing classically orthogonality is suggested. First, to fix some terminology, let us note that classically $\sim(\mathrm{Con}(l,m)\,\&\,\mathrm{Unort}(l,m))$ is equivalent to $\mathrm{Par}(l,m)\vee\mathrm{Ort}(l,m)$. This latter we call the *nonobliqueness* of lines l and m. (Should *OALD* ever list nonoblique, my suggestion to them is 'horizontal or vertical'!). The classical axioms are:

CLO1. Incompatibility of parallelism and orthogonality

$$\vdash\ \sim(\mathrm{Par}(l,m)\,\&\,\mathrm{Ort}(l,m)).$$

CLO2. Transitivity of nonobliqueness

$$\vdash(\mathrm{Par}(l,m)\vee\mathrm{Ort}(l,m))\,\&\,(\mathrm{Par}(l,n)\vee\mathrm{Ort}(l,n))\ \to\ \mathrm{Par}(m,n)\vee\mathrm{Ort}(m,n).$$

CLO3. Uniqueness axiom for orthogonality

$$\vdash\mathrm{Ort}(l,m)\,\&\,\mathrm{Ort}(l,n)\ \to\ \mathrm{Par}(m,n).$$

In case $\mathrm{Inc}(a, m\cdot n)$, the last axiom gives by Lemma 7.3 $\mathrm{Ort}(l, m\cdot n)\to\mathrm{EqLn}(m,n)$.

It is quite natural to think, in classical geometry, in terms of the equivalence relation nonobliqueness and the exclusion principle CLO1. Let us show that the above axioms give the usual properties of orthogonality, listing also a couple of previously encountered principles for reference:

Theorem 8.7. (o) $\sim\mathrm{Ort}(l,l)$

(i) $\mathrm{Par}(l,m)\,\&\,\mathrm{Par}(l,n)\to\mathrm{Par}(m,n)$,

(ii) $\mathrm{Par}(l,m)\,\&\,\mathrm{Ort}(l,n)\to\mathrm{Ort}(m,n)$,

(iii) $\mathrm{Ort}(l,m)\,\&\,\mathrm{Par}(l,n)\to\mathrm{Ort}(m,n)$,

(iv) $\mathrm{Ort}(l,m)\,\&\,\mathrm{Ort}(l,n)\to\mathrm{Par}(m,n)$.

Proof. (o) By CLO1, $\mathrm{Par}(l,l)$ implies $\sim\mathrm{Ort}(l,l)$. (i) is transitivity of parallelism. (ii) Assume $\mathrm{Par}(l,m)$, $\mathrm{Ort}(l,n)$. Then $(\mathrm{Par}(l,m)\vee\mathrm{Ort}(l,m))\,\&\,(\mathrm{Par}(l,n)\vee\mathrm{Ort}(l,n))$, so by axiom CLO2, $\mathrm{Par}(m,n)\vee\mathrm{Ort}(m,n)$. If $\mathrm{Par}(m,n)$, $\mathrm{Par}(l,m)$ gives $\mathrm{Par}(l,n)$. But by CLO1, the assumption $\mathrm{Ort}(l,n)$ implies $\sim\mathrm{Par}(l,n)$, so that $\sim\mathrm{Par}(m,n)$. Therefore $\mathrm{Ort}(m,n)$. (iii) is similar to the previous. (iv) is uniqueness of orthogonals CLO3. □

If in the results (i)–(iv) we weaken the consequents into $\mathrm{Par}(m,n)\vee\mathrm{Ort}(m,n)$, the disjunction of the four antecedents is classically equivalent to the antecedent of the transitivity axiom for nonobliqueness (by the equivalence of $(A\to C)\,\&\,(B\to C)$ and $A\vee B\to C$). This may explain why axiom CLO2 does not seem to have appeared in previous classical literature.

The constructive forms of the principles (i)–(iv) are

$$\mathrm{Con}(l, m) \to \mathrm{Con}(l, n) \vee \mathrm{Con}(m, n),$$

$$\mathrm{Unort}(l, m) \to \mathrm{Con}(l, n) \vee \mathrm{Unort}(m, n),$$

$$\mathrm{Unort}(l, m) \to \mathrm{Unort}(l, n) \vee \mathrm{Con}(m, n),$$

$$\mathrm{Con}(l, m) \to \mathrm{Unort}(l, n) \vee \mathrm{Unort}(m, n).$$

The first is the familiar apartness axiom for convergence, the second is proved in Theorem 8.3. The third follows by symmetry from previous. The fourth is proved in Theorem 8.6. In the classical case, the distributivity laws for conjunction and disjunction lead from (i)–(iv) to CLO2. But constructively we cannot derive O2 from the above four constructive principles in that way.

Let us now see to it that the parallel and orthogonal line constructions have the right relation to each other. I show that the parallel line construction can be effected in terms of the orthogonality construction, by proving that

$$\mathrm{ort}(\mathrm{ort}(l, a), a) : \mathrm{Line}$$

has the properties of the function par of Section 7, namely

$$\mathrm{Par}(l, \mathrm{ort}(\mathrm{ort}(l, a), a)),$$

$$\mathrm{Inc}(a, \mathrm{ort}(\mathrm{ort}(l, a), a)).$$

The latter is immediate from O3. To prove the former we need the

Lemma 8.8. *Assume* $\mathrm{Inc}(b, \mathrm{ort}(l, a))$. *Then*

$$\mathrm{EqLn}(\mathrm{ort}(l, a), \mathrm{ort}(l, b)).$$

Proof. The substitution $[b/a, \mathrm{ort}(l, a)/l, \mathrm{ort}(l, b)/m, l/n]$ in the contraposition of the uniqueness axiom O4 gives

$$\mathrm{Inc}(b, \mathrm{ort}(l, a) \cdot \mathrm{ort}(l, b)) \,\&\, \mathrm{Ort}(\mathrm{ort}(l, a) \cdot \mathrm{ort}(l, b), l) \to \mathrm{EqLn}(\mathrm{ort}(l, a), \mathrm{ort}(l, b)).$$

All the assumptions of the antecedent are satisfied which proves the lemma. □

Theorem 8.9. $\mathrm{Par}(l, \mathrm{ort}(\mathrm{ort}(l, a), a))$.

Proof. Assume $\mathrm{Con}(l, \mathrm{ort}(\mathrm{ort}(l, a), a))$. Make the abbreviation

$$\mathrm{pt}(l, \mathrm{ort}(\mathrm{ort}(l, a), a)) = b : \mathrm{Point}.$$

We have $\mathrm{Inc}(b, l)$ and $\mathrm{Inc}(b, \mathrm{ort}(\mathrm{ort}(l, a), a))$ by the intersection point axiom II. By the lemma,

$$\mathrm{EqLn}(\mathrm{ort}(\mathrm{ort}(l, a), a), \mathrm{ort}(\mathrm{ort}(l, a), b)).$$

Since convergence implies apartness (Theorem 3.2), we have DiLn(l, ort(ort(l, a), a)) from the assumption. We get by substitution of equai lines DiLn(l, ort(ort(l, a), b)). By O4.

$$\mathrm{Apt}(b, l \mid \mathrm{ort}(\mathrm{ort}(l, a), b)) \vee \mathrm{Unort}(l \mid \mathrm{ort}(\mathrm{ort}(l, a), b), \mathrm{ort}(l, a)).$$

But all of the disjuncts are impossible. Therefore Par(l, ort(ort(l, a), a)). □

By the uniqueness of parallels (Theorem 7.1) we arrive at

Corollary 8.10. EqLn(par(l, a), ort(ort(l, a), a)).

Orthogonal geometry can be extended to Euclidean geometry in various ways. Line segments on parallel lines can be compared in affine geometry, as in the second example of the next section where a *translation* is defined. If a *rotation* is added, arbitrary line segments can be compared as is characteristic of Euclidean geometry. We can postulate a construction rule for rotating a point around a second point, in the direction given by a third point. To axiomatize its properties, a way is needed for expressing that the constructed point and the third point are in the same direction from the second point. Various choices are possible, but we shall not pursue the matter further here.

9. Solution of geometric problems

In this section, I will illustrate by examples how geometric problems are solved in constructive geometry, and also introduce concepts and notation that will be used in the type-theoretical formalization of Section 11.

The form given to geometric problems as well as the terminology in Section 1 comes from Mäenpää [6, Section 3.1]. In type-theoretical notation, we have the form

$$(\forall x : A)(\exists y : B(x))C(x, y).$$

Mäenpää notes that this form is the same as the one encountered in programming problems. Geometric problems can be seen as particular cases: Their solution is a function or program that converts any given data $a : A$ into some $b : B(a)$ and a proof that $C(a, b)$.

Problem 9.1. Assume orthogonal geometry. Given a point and a line, to find a point incident with the line.

Formalization. We shall formalize the problem as

$$(\forall x : \mathrm{Point})(\forall y : \mathrm{Line})(\exists z : \mathrm{Point})\mathrm{Inc}(z, y).$$

We also make the abbreviations Pt = Point and Ln = Line. The derivation is written in tree form.

Solution:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1.}{l:\mathrm{Ln}} \quad \dfrac{\overset{1.}{l:\mathrm{Ln}} \quad \overset{2.}{a:\mathrm{Pt}}}{\mathrm{ort}(l,a):\mathrm{Ln}}\,\mathrm{ort}}{}\quad \dfrac{\dfrac{\overset{1.}{l:\mathrm{Ln}} \quad \dfrac{\overset{1.}{l:\mathrm{Ln}} \quad \overset{2.}{a:\mathrm{Pt}}}{\mathrm{ort}(l,a):\mathrm{Ln}}\,\mathrm{ort}}{\vdash \mathrm{Con}(l,\mathrm{ort}(l,a)) \vee \mathrm{Unort}(l,\mathrm{ort}(l,a))}\,\mathrm{O1} \quad \dfrac{\dfrac{\overset{1.}{l:\mathrm{Ln}} \quad \overset{2.}{a:\mathrm{Pt}}}{\mathrm{ort}(l,a):\mathrm{Ln}}\,\mathrm{ort}}{\vdash \mathrm{Ort}(l,\mathrm{ort}(l,a))}\,\mathrm{O3}}{\vdash \mathrm{Con}(l,\mathrm{ort}(l,a))}\,\mathrm{mtp}}{\mathrm{pt}(l,\mathrm{ort}(l,a)):\mathrm{Pt}}\,\mathrm{pt}}{\vdash \mathrm{Inc}(\mathrm{pt}(l,\mathrm{ort}(l,a)),l)}\,\text{Inc-ax}}{}}{}\quad}{}}{}
$$

$$
\dfrac{\dfrac{\dfrac{M \qquad \vdash \mathrm{Inc}(\mathrm{pt}(l,\mathrm{ort}(l,a)),l)}{\vdash (\exists z:\mathrm{Pt})\,\mathrm{Inc}(z,l)}\,\exists\text{-intr}}{\vdash (\forall y:\mathrm{Ln})(\exists z:\mathrm{Pt})\,\mathrm{Inc}(z,y)}\,\forall\text{-intr, 1.}}{\vdash (\forall x:\mathrm{Pt})(\forall y:\mathrm{Ln})(\exists z:\mathrm{Pt})\,\mathrm{Inc}(z,y)}\,\forall\text{-intr, 2.}
$$

(In the $\exists$-introduction, the letter M abbreviates the derivation of $\mathrm{pt}(l, \mathrm{ort}(l, a))$: Pt that was done just above. mtp stands for the inference, from $A \vee B$ and $\sim B$ to infer A.) The solution seems not to involve the point x. Actually that is not the case, but the dependence of z on x through the proof object for $\mathrm{Con}(l, (\mathrm{ort}(l, a)))$ cannot be represented in traditional logical notations such as the predicate calculus. For giving the dependence, one needs the formalism of type theory. When formalized in type theory, the solution to the above problem is a program that takes an arbitrary point and a line as arguments, and gives as value a point incident with the line, together with a proof of incidence.

Viewed as a programming system, type theory is able to express both programs and their specifications, or what tasks the programs execute. In fact, the basic form of judgment of type theory, $a : A$, has a variety of readings:

a is an element of set A,

a is a proof of proposition A,

a is a program that meets specification A.

The last reading can be seen as a special case of the following one: a is a solution to problem A. Type theory can also be used for the expression of geometric problems and their solutions. The formalization of constructive geometry in type theory will at the same time work as its computer implementation, with the property that if we have a derivation of a judgment of the form $a : A$, it is automatically checked for correctness.

Let us look at a second example of a problem. Even though we do not have the concept of equidistance of two pairs of points (or congruence of line segments) in affine geometry, a construction that in fact moves finite line segments, can be effected under suitable assumptions. In type theory, the assumptions are listed as a *context*,

according to the following model (here with assertion signs instead of proof objects):

$$(\ a, b, c\colon \mathrm{Point}, \vdash \mathrm{DiPt}(a, b), \vdash \mathrm{Apt}(c, \ln(a, b)) \ ).$$

Let us call this context Triangle Figure. We aim at establishing what is naturally called a Parallelogram Figure:

$$(\ a, b, c, d\colon \mathrm{Point}, \vdash \mathrm{DiPt}(a, b), \vdash \mathrm{DiPt}(c, d), \vdash \mathrm{DiPt}(a \cdot b, c \cdot d),$$

$$l, m, n, r\colon \mathrm{Line}, \vdash \mathrm{DiLn}(l, m), \vdash \mathrm{DiLn}(n, r), \vdash \mathrm{Par}(l, m), \vdash \mathrm{Par}(n, r),$$

$$\vdash \mathrm{Inc}(a \cdot b, l), \vdash \mathrm{Inc}(c \cdot d, m), \vdash \mathrm{Inc}(a \cdot c, n), \vdash \mathrm{Inc}(b \cdot d, r) \ ).$$

It is easy to see that such a figure cannot be degenerate. From $\mathrm{DiPt}(a, c)$ and $\mathrm{DiLn}(l, m)$ we infer by axiom III that $\mathrm{Apt}(a \mid c, l \mid m)$. This leaves the cases $\mathrm{Apt}(a, m)$ and $\mathrm{Apt}(c, l)$ both of which imply that the parallelogram cannot be flat. In classical geometry, the corresponding assumption is that the diagonals of a parallelogram intersect (the axiom of Fano).

Instead of the tree-form derivation of the previous example I shall reason more informally, by manipulating contexts according to the rules available.

Problem 9.2. Assume affine geometry. Given a Triangle Figure, to find a Parallelogram Figure.

Solution. I will solve the problem by constructing a mapping from the context Triangle Figure to an instance of the context Parallelogram Figure.

Begin with the context $(\ a, b, c\colon \mathrm{Point}, \ \vdash \mathrm{DiPt}(a, b), \vdash \mathrm{Apt}(c, \ln(a, b)) \ )$. From $\mathrm{Apt}(c, \ln(a, b))$ we get by Lemma 4.3 $\mathrm{DiPt}(a, c)$, so we may construct $\ln(a, c)\colon \mathrm{Line}$. Next construct $\mathrm{par}(\ln(a, b), c)\colon \mathrm{Line}$ and $\mathrm{par}(\ln(a, c), b)\colon \mathrm{Line}$. We have $\mathrm{Con}(\ln(a, b), \ln(a, c))$ and $\mathrm{Par}(\ln(a, b), \mathrm{par}(\ln(a, b), c))$, so that by compatibility $\mathrm{Con}(\ln(a, c), \mathrm{par}(\ln(a, b), c))$ follows. Since $\mathrm{Par}(\ln(a, c), \mathrm{par}(\ln(a, c), b))$, we also have $\mathrm{Con}(\mathrm{par}(\ln(a, b), c), \mathrm{par}(\ln(a, c), b))$. Therefore we may construct the point $\mathrm{pt}(\mathrm{par}(\ln(a, b), c), \mathrm{par}(\ln(a,c), b))\colon \mathrm{Point}$. Abbreviate this to $d\colon \mathrm{Point}$. We have $a, b, c\colon \mathrm{Point}$ by assumption, and $d\colon \mathrm{Point}$ by construction. Further, by assumption $\mathrm{DiPt}(a, b)$, and it is easy to verify $\mathrm{DiPt}(c, d)$ and $\mathrm{DiPt}(a \cdot b, c \cdot d)$. Next, we have $\ln(a, b), \ln(a, c)\colon \mathrm{Line}$ and $\mathrm{par}(\ln(a, b), c), \mathrm{par}(\ln(a, c), b)\colon \mathrm{Line}$. $\mathrm{Par}(\ln(a, b), \mathrm{par}(\ln(a, b), c))$ and $\mathrm{Par}(\ln(a, c), \mathrm{par}(\ln(a, c), b))$ are instances of the axiom for constructed parallels. The incidences $\mathrm{Inc}(a \cdot b, \ln(a, b))$, $\mathrm{Inc}(c \cdot d, \mathrm{par}(\ln(a, b), c))$, $\mathrm{Inc}(a \cdot c, \ln(a, c))$ and $\mathrm{Inc}(b \cdot d, \mathrm{par}(\ln(a, c), b))$ also are easy to verify. Putting all this together, we have the context

$$(\ a, b, c, d\colon \mathrm{Point}, \vdash \mathrm{DiPt}(a, b), \vdash \mathrm{DiPt}(c, d), \vdash \mathrm{DiPt}(a \cdot b, c \cdot d),$$

$$\ln(a, b), \ln(a, c), \mathrm{par}(\ln(a, b), c), \mathrm{par}(\ln(a, c), b)\colon \mathrm{Line},$$

$$\vdash \mathrm{Par}(\ln(a, b), \mathrm{par}(\ln(a, b), c)), \vdash \mathrm{Par}(\ln(a, c), \mathrm{par}(\ln(a, c), b)),$$

$\vdash \mathrm{Inc}(a \cdot b, \ln(a, b)), \vdash \mathrm{Inc}(c \cdot d, \mathrm{par}(\ln(a, b), c)),$

$\vdash \mathrm{Inc}(a \cdot c, \ln(a, c)), \vdash \mathrm{Inc}(b \cdot d, \mathrm{par}(\ln(a, c), b))$).

We thus have a method for transforming the context Triangle Figure into a context Parallelogram Figure. It consists in simply substituting the three parameters of the given triangle context for a, b, and c in the above parallelogram context. Specifically, the point sought d is the value of $\mathrm{pt}(\mathrm{par}(\ln(a, b), c), \mathrm{par}(\ln(a, c), b))$. A line segment, as determined by two distinct points a and b, can be moved to a point c on a line parallel to $\ln(a, b)$ by computing the value of d. (Comparability of line segments along parallel lines is a characteristic of affine geometry.) The solution to Problem 9.2 defines a *translation* of a line segment from one line to another that is parallel to it.

From the above parallelogram construction, a general way of looking at solutions of geometric problems is suggested: The solutions are mappings from the *data context* of the problem into its *goal context*. We can give explicit expressions to these mappings in type theory, as functions between contexts.

As a third problem, I will construct a *projectivity* in projective geometry. It means the following: We have two distinct lines l and m, with three pairwise distinct points a_1, a_2, a_3 incident with l, and b_1, b_2, b_3 incident with m. To exclude simplifying special cases, we assume these points to be distinct from the intersection point of l and m. A *perspectivity* between a_1, a_2, a_3 and b_1, b_2, b_3 obtains if there is a point c such that each of a_1, a_2, a_3 is incident with a line through c and one of b_1, b_2, b_3 and the other way around. A *projectivity* between a_1, a_2, a_3 and b_1, b_2, b_3 obtains if there is a finite sequence of perspectivities starting with a_1, a_2, a_3 and ending with b_1, b_2, b_3 (or the other way around). It will turn out that a sequence of two perspectivities is enough.

Problem 9.3. Assume projective geometry. Given three pairwise distinct points on a line and another three on another line, with all the six points distinct from the intersection of the lines, to find a projectivity between the points of the first and second lines.

Formalization. Define *collinearity* of three points, and *concurrency* of three lines, as follows:

$$\mathrm{Coll}(x, y, z) = (\exists v : \mathrm{Ln})\mathrm{Inc}(x \cdot y \cdot z, v) \text{ where } x, y, z : \mathrm{Pt},$$

$$\mathrm{Conc}(x, y, z) = (\exists v : \mathrm{Pt})\mathrm{Inc}(v, x \cdot y \cdot z), \text{ where } x, y, z : \mathrm{Ln}.$$

Next define *perspectivity* of six points by

$$\begin{aligned}\mathrm{Pers}(x_1, x_2, x_3, y_1, y_2, y_3) = (\exists v_1, v_2, v_3 : \mathrm{Ln})(&\mathrm{Coll}(x_1, x_2, x_3) \,\&\, \mathrm{Coll}(y_1, y_2, y_3)\\ &\&\mathrm{Conc}(v_1, v_2, v_3)\\ &\&\, \mathrm{DiLn}(v_1, v_2) \,\&\, \mathrm{DiLn}(v_1 \cdot v_2, v_3)\\ &\&\mathrm{Inc}(x_1 \cdot y_1, v_1) \,\&\, \mathrm{Inc}(x_2 \cdot y_2, v_2) \,\&\, \mathrm{Inc}(x_3 \cdot y_3, v_3)).\end{aligned}$$

Note that the definition is general, it permits degeneracies whereas in the present problem we assume all the points distinct and so on, in order to deal directly with the most typical situation.

Next define *projectivity* of six points by

$$\mathrm{Proj}(x_1, x_2, x_3, z_1, z_2, z_3) = (\exists y_1, y_2, y_3 : \mathrm{Pt})(\mathrm{Pers}(x_1, x_2, x_3, y_1, y_2, y_3) \\ \& \mathrm{Pers}(y_1, y_2, y_3, z_1, z_2, z_3)).$$

The problem is now formalized as follows: From the data context

$$(a_1, a_2, a_3, b_1, b_2, b_3 : \mathrm{Pt}, l, m : \mathrm{Ln},$$

$$\vdash \mathrm{DiLn}(l, m), \vdash \mathrm{DiPt}(a_1, a_2), \vdash \mathrm{DiPt}(a_3, a_1 \cdot a_2),$$

$$\vdash \mathrm{DiPt}(a_1 \cdot a_2 \cdot a_3, \mathrm{pt}(l, m)), \vdash \mathrm{DiPt}(b_1, b_2), \vdash \mathrm{DiPt}(b_3, b_1 \cdot b_2),$$

$$\vdash \mathrm{DiPt}(b_1 \cdot b_2 \cdot b_3, \mathrm{pt}(l, m)),$$

$$\vdash \mathrm{Inc}(a_1 \cdot a_2 \cdot a_3, l), \vdash \mathrm{Inc}(b_1 \cdot b_2 \cdot b_3, m) \ )$$

to derive $\mathrm{Proj}(b_1, b_2, b_3, a_1, a_2, a_3)$.

Solution. The following lemmas are needed for the construction:

$$\vdash \mathrm{Apt}(a_1 \cdot a_2 \cdot a_3, m),$$

$$\vdash \mathrm{Apt}(b_1 \cdot b_2 \cdot b_3, l),$$

$$\vdash \mathrm{DiPt}(a_1 \cdot a_2 \cdot a_3, b_1 \cdot b_2 \cdot b_3).$$

The first two follow from axiom III. By $\mathrm{DiPt}(a_1, \mathrm{pt}(l, m))$ and $\mathrm{Inc}(\mathrm{pt}(l, m), m)$ we get $\mathrm{Apt}(a_1, m)$, and so on. The third one follows straight from Theorem 12.1.

Now to the construction proper: First construct $\mathrm{ln}(a_2, a_3)$, $\mathrm{ln}(a_2, b_2)$, $\mathrm{ln}(a_3, b_3)$. Since $\mathrm{EqLn}(l, \mathrm{ln}(a_2, a_3)$ by Lemma 4.5 and $\mathrm{Apt}(b_3, l)$ by above, $\mathrm{Apt}(b_3, \mathrm{ln}(a_2, a_3))$. By the triangle axiom (Theorem 4.4), $\mathrm{Apt}(a_2, \mathrm{ln}(a_3, b_3))$. Therefore $\mathrm{DiLn}(\mathrm{ln}(a_2, b_2), \mathrm{ln}(a_3, b_3))$. By the projective axiom, $\mathrm{Con}(\mathrm{ln}(a_2, b_2), \mathrm{ln}(a_3, b_3))$. Construct $\mathrm{pt}(\mathrm{ln}(a_2, b_2), \mathrm{ln}(a_3, b_3))$. Let $\mathrm{pt}(\mathrm{ln}(a_2, b_2), \mathrm{ln}(a_3, b_3)) = p_1 : \mathrm{Pt}$. Construct $\mathrm{ln}(b_1, b_2)$. Then $\mathrm{EqLn}(\mathrm{ln}(b_1, b_2), m)$. Since $\mathrm{Apt}(a_2, m)$, $\mathrm{Apt}(a_2, \mathrm{ln}(b_1, b_2))$. By the triangle axiom, $\mathrm{Apt}(b_1, \mathrm{ln}(a_2, b_2))$. Construct $\mathrm{ln}(b_1, a_3)$. Then $\mathrm{DiLn}(\mathrm{ln}(a_2, b_2), \mathrm{ln}(b_1, a_3))$. Construct $\mathrm{pt}(\mathrm{ln}(a_2, b_2), \mathrm{ln}(b_1, a_3)) = p_2 : \mathrm{Pt}$. Since $\mathrm{Apt}(a_3, m)$ and $\mathrm{EqLn}(m, \mathrm{ln}(b_1, b_3))$, $\mathrm{Apt}(a_3, \mathrm{ln}(b_1, b_3))$, so again $\mathrm{Apt}(b_1, \mathrm{ln}(a_3, b_3))$. Since $\mathrm{Inc}(p_1, \mathrm{ln}(a_3, b_3))$, $\mathrm{DiPt}(b_1, p_1)$. Construct $\mathrm{ln}(b_1, p_1)$.

By the above construction, we have:

$\mathrm{Inc}(b_1 \cdot b_2 \cdot b_3, m)$, so that $(\exists v : \mathrm{Ln})\mathrm{Inc}(b_1 \cdot b_2 \cdot b_3, v)$. Therefore $\mathrm{Coll}(b_1, b_2, b_3)$.

$\mathrm{Inc}(b_1 \cdot p_2 \cdot a_3, \mathrm{ln}(b_1, a_3))$, so that $\mathrm{Coll}(b_1, p_2, a_3)$.

$\mathrm{Inc}(p_1, (\mathrm{ln}(b_1, p_1) \cdot \mathrm{ln}(a_2, b_2) \cdot \mathrm{ln}(a_3, b_3))$, so that

$\mathrm{Conc}(\mathrm{ln}(b_1, p_1), \mathrm{ln}(a_2, b_2), \mathrm{ln}(a_3, b_3))$.

$\mathrm{DiLn}(\ln(b_1, p_1), \ln(a_2, b_2))$.

$\mathrm{DiLn}(\ln(b_1, p_1) \cdot \ln(a_2, b_2), \ln(a_3, b_3))$.

$\mathrm{Inc}(b_1 \cdot b_1, \ln(b_1, p_1)) \,\&\, \mathrm{Inc}(b_2 \cdot p_2, \ln(a_2, b_2)) \,\&\, \mathrm{Inc}(b_3 \cdot a_3, \ln(a_3, b_3))$.

Therefore $(\exists v_1, v_2, v_3 : \mathrm{Ln})(\mathrm{Coll}(b_1, b_2, b_3) \,\&\, \mathrm{Coll}(b_1, p_2, a_3) \,\&\, \mathrm{Conc}(v_1, v_2, v_3)$

$\&\mathrm{DiLn}(v_1, v_2)\&\mathrm{DiLn}(v_1 \cdot v_2, v_3)$

$\&\, \mathrm{Inc}(b_1 \cdot b_1, v_1) \,\&\, \mathrm{Inc}(b_2 \cdot p_2, v_2) \,\&\, \mathrm{Inc}(b_3 \cdot a_3, v_3))$, so that

$\mathrm{Pers}(b_1, b_2, b_3, b_1, p_2, a_3)$.

Next construct $\ln(a_1, b_1)$, $\ln(a_1, a_2)$. Since $\mathrm{Apt}(b_2, l)$ and $\mathrm{EqLn}(l, \ln(a_1, a_2))$, $\mathrm{Apt}(b_2, \ln(a_1, a_2))$. By the triangle axiom, $\mathrm{Apt}(a_1, \ln(a_2, b_2))$. Therefore $\mathrm{DiLn}(\ln(a_1, b_1), \ln(a_2, b_2))$. Construct $\mathrm{pt}(\ln(a_1, b_1), \ln(a_2, b_2)) = p_3 : \mathrm{Pt}$. Similarly, $\mathrm{Apt}(a_3, \ln(a_2, b_2))$, so from $\mathrm{Inc}(p_3, \ln(a_2, b_2))$ we get by Theorem 12.1 that $\mathrm{DiPt}(a_3, p_3)$. Construct $\ln(a_3, p_3)$.

By the above construction, we have:

$\mathrm{Inc}(b_1 \cdot p_2 \cdot a_3, \ln(b_1, a_3))$, so that $\mathrm{Coll}(b_1, p_2, a_3)$.

$\mathrm{Inc}(a_1 \cdot a_2 \cdot a_3, l)$, so that $\mathrm{Coll}(a_1, a_2, a_3)$.

$\mathrm{Inc}(p_3, (\ln(a_1, b_1) \cdot \ln(a_2, b_2) \cdot \ln(a_3, p_3))$, so that

$\mathrm{Conc}(\ln(a_1, b_1), \ln(a_2, b_2), \ln(a_3, p_3))$.

$\mathrm{DiLn}(\ln(a_1, b_1), \ln(a_2, b_2))$.

$\mathrm{DiLn}(\ln(a_1, b_1) \cdot \ln(a_2, b_2), \ln(a_3, p_3))$.

$\mathrm{Inc}(b_1 \cdot a_1, \ln(a_1, b_1)) \,\&\, \mathrm{Inc}(p_2 \cdot a_2, \ln(a_2, b_2)) \,\&\, \mathrm{Inc}(a_3 \cdot a_3, \ln(a_3, p_3))$.

These together imply $\mathrm{Pers}(b_1, p_2, a_3, a_1, a_2, a_3)$.

From $\mathrm{Pers}(b_1, b_2, b_3, b_1, p_2, a_3) \,\&\, \mathrm{Pers}(b_1, p_2, a_3, a_1, a_2, a_3)$ we infer finally $\mathrm{Proj}(b_1, b_2, b_3, a_1, a_2, a_3)$, which solves Problem 9.3.

Once proof objects are added, problems and theorems merge into one concept. In the above, we proved the proposition $\mathrm{Proj}(b_1, b_2, b_3, a_1, a_2, a_3)$. But on the other hand, as becomes apparent in Section 11 the proof converts at once into a solution of the problem of constructing an object of an appropriate type, namely the one expressed by the proposition $\mathrm{Proj}(b_1, b_2, b_3, a_1, a_2, a_3)$. The difference between problems and theorems is, as mentioned in the opening section, that sometimes the sought object can be empty or absent and we have a theorem of form $(\forall x : A)C(x)$, or the object is of no interest in itself (save that it exists) and we just think of $(\forall x : A)(\exists y : B(x))C(x, y)$ as a theorem rather than a problem.

A projectivity between two lines defines a *projective transformation* from points of one line to points of the other. If we take an arbitrary $x : \mathrm{Pt}$ such that $\mathrm{Inc}(x, m)$,

construct $\mathrm{ln}(x, p_1)$, then $\mathrm{pt}(\mathrm{ln}(x, p_1), \mathrm{ln}(b_1, a_3))$, then $\mathrm{ln}(p_3, \mathrm{pt}(\mathrm{ln}(x, p_1), \mathrm{ln}(b_1, a_3)))$, and finally

$$\mathrm{pt}(l, \mathrm{ln}(p_3, \mathrm{pt}(\mathrm{ln}(x, p_1), \mathrm{ln}(b_1, a_3)))),$$

we obtain the projective transformation of x as defined by the projectivity $\mathrm{Proj}(b_1, b_2, b_3, a_1, a_2, a_3)$. (Naturally, the correctness of the above construction steps has to be shown.) The full expression for the function that performs the projective transformation from line m to line l is

$$\mathrm{pt}(l, \mathrm{ln}(\mathrm{pt}(\mathrm{ln}(a_1, b_1), \mathrm{ln}(a_2, b_2), \mathrm{pt}(\mathrm{ln}(x, \mathrm{pt}(\mathrm{ln}(a_2, b_2), \mathrm{ln}(a_3, b_3))), \mathrm{ln}(b_1, a_3)))).$$

such expressions are of course rather unwieldy to read. They are the result of coding linearly, by the device of parentheses, the generation of objects in a derivation that has the form of a two-dimensional tree.

A final remark: In the above, collinearity was defined as the existence of a line with which the points are incident. If we start with a constructively basic relation, we can define three points a, b, c to be *planar* through the condition $(\forall x : \mathrm{Ln})\mathrm{Apt}(a \mid b \mid c, x)$. A negative notion of collinearity, slightly weaker than the one we used, is obtained by denying a, b, c to be planar. If any two of the points a, b, c are distinct, the conditions become equivalent.

10. Existence

One starts a geometric construction from a blank paper or computer screen. Existence is derived from construction, or else appears as a hypothesis. In the classical axiomatizations one postulates, instead, what could be written in the present notation as

$$\sim \mathrm{EqPt}(a, b) \rightarrow (\exists x : \mathrm{Line})\mathrm{Inc}(a \cdot b, x),$$

$$\sim \mathrm{Par}(l, m) \rightarrow (\exists x : \mathrm{Point})\mathrm{Inc}(x, l \cdot m).$$

If we reformulate the antecedents in positive terms, the two axioms are derivable in any context that contains $\vdash \mathrm{DiPt}(a, b)$ (resp. $\vdash \mathrm{Con}(l, m)$), by the use of the axioms for constructed lines and points II. The existence of a line, for example, is derived as follows:

$$\dfrac{\dfrac{a,b : \mathrm{Pt} \quad \vdash \mathrm{DiPt}(a,b)}{\mathrm{ln}(a,b) : \mathrm{Ln}}\,\mathrm{ln} \qquad \dfrac{\vdash \mathrm{DiPt}(a,b) \quad \vdash \mathrm{DiPt}(a,b) \rightarrow \mathrm{Inc}(a \cdot b, \mathrm{ln}(a,b))}{\vdash \mathrm{Inc}(a \cdot b, \mathrm{ln}(a,b))}\,\text{modus ponens}}{\vdash (\exists x : \mathrm{Ln})\mathrm{Inc}(a \cdot b, x)}\,\exists\text{-intr}$$

The classical axiomatizations also postulate existence axioms of a different kind, ones that cannot be derived simply in the present constructive axiomatization.

A typical example is the axiom

$$(\forall x:\mathrm{Line})(\exists y, z:\mathrm{Point})(\sim \mathrm{EqPt}(y, z)\,\&\,\mathrm{Inc}(y\cdot z, x)).$$

But from an arbitrary line l we only know that l: Line. Obviously, none of our constructions are applicable, and no other objects can be derived. The constructive reinterpretation of the classical existence axioms can be effected as follows.

An *open context* is one whose assumptions declare points $a, b, c, \ldots$: Point and lines $l, m, n, \ldots$: Line and their properties in terms of the basic relations DiPt, DiLn, Con, Apt and Unort. In these terms, the solution to Problem 9.1 can be seen as a reduction of the context

$$(\ l:\mathrm{Line},\ \vdash(\exists x:\mathrm{Point})\mathrm{Inc}(x, l)\ )$$

to the open context (a: Point, l: Line). More generally, if a context with existence assumptions can be reduced to the basic objects and their basic properties, it is given a finitary sense as explained in Section 2 (finitary relative to the concepts Point and Line that in themselves are ideal). Note that all of our basic concepts and constructions are continuous in an obvious sense in their arguments, whereas this is not the case for the properties of constructed objects (incidence, parallelism and orthogonality in axioms I1, A1 and O3).

In the present axiomatization of constructive geometry, all existence is reduced to existence in the primary sense of type theory, as expressed by the basic form of judgment $a: A$. This constructive geometry does not stipulate what the basic objects are, or how their basic relations are proved. In this sense it belongs to abstract mathematics, rather than to traditional constructive mathematics, where the aim has been to define once and for all the natural numbers and build all other mathematical structures upon them.

11. Definition of constructive geometry in type theory

In this section, we shall first formalize the geometrical axioms in terms of higher-level type theory. (See [9] for higher-level type theory.) Type theory itself has been implemented in Gothenburg, in what is known as the ALF system. (See [7] for ALF.) The notation of ALF is a variant of that of type theory. We give a formalization of the different geometries in ALF also. As a consequence the latter formalization of the axioms of constructive geometry also works directly as the ALF code for the implementation of geometry. The solving of a geometric problem can be done as proof-editing in the ALF system. One starts by the choice of the appropriate geometry and the declaration of what is given in the problem as a context. Next the specification of the problem is given and its solution constructed interactively, with the system guiding the construction through type checking. When the process is finished, one has a solution checked for correctness. It is at the same time a program in ALF that takes the given of the problem as argument, and returns as value a correct solution.

Let us first formalize the geometrical axioms in purely type-theoretical notation. No logical constants appear in these formalizations. Conjunction, disjunction and negation are instead formalized after the pattern of the corresponding elimination rules of natural deduction. To enhance the readability of the formal expressions, we use the following definitions.

$$\mathrm{Et} = (A:\mathrm{Set})(B:\mathrm{Set})(C:\mathrm{Set})((A)(B)C)C:\mathrm{Type},$$

$$\mathrm{Vel} = (A:\mathrm{Set})(B:\mathrm{Set})(C:\mathrm{Set})((A)C)((B)C)C:\mathrm{Type},$$

$$\mathrm{Non} = (A:\mathrm{Set})(B:\mathrm{Set})(A)B:\mathrm{Type},$$

to be called *type conjunction, type disjunction* and *type negation*, respectively. The existential quantifier can be expressed in a similar manner, but we shall not use it in the present axiomatization of geometry. It tends to shorten the formal proofs of our theorems considerably if the geometrical axioms and inferences are formulated without the usual logical constants. For example, we declare a geometrical function irr_dipt, typed as

$$\mathrm{irr\_dipt}:(a:\mathrm{Point})\mathrm{Non}(\mathrm{DiPt}(a,a)).$$

Given a: Point and an arbitrary proposition B, irr_dipt applies as

$$\mathrm{irr\_dipt}(a,B):(\mathrm{DiPt}(a,a))B.$$

If we had a proof $w:\mathrm{DiPt}(a,a)$, we could turn it into a proof of any proposition by irr_dipt. Instead of declaring logical constants we can construct objects of the appropriate types. For example, the first rule of disjunction introduction requires an object of the type $(A)\mathrm{Vel}(A,B)$, given the context (A:Set, B:Set). Let C:Set, $x:A$, $y:(A)C$ and $z:(B)C$. Then $y(x):C$, so that

$$(z)y(x):((B)C)C,$$

$$(y)(z)y(x):((A)C)((B)C)C,$$

$$(C)(y)(z)y(x):(C:\mathrm{Set})((A)C)((B)C)C,$$

$$(x)(C)(y)(z)y(x):(A)(C:\mathrm{Set})((A)C)((B)C)C.$$

Thus, we have proved the type-logical form of the disjunction introduction rule in the context (A:Set, B:Set). The other rules are equally simple to prove. Some of the familiar patterns of logical inference also take on a new shape. We started in Section 3 by proving the symmetry of the apartness relation DiPt, by the use of the scheme, from $\sim A$ and $A \vee B$ to infer B (modus tollendo ponens). The form of this rule is

$$(A:\mathrm{Set})(B:\mathrm{Set})((C:\mathrm{Set})(A)C)((D:\mathrm{Set})((A)D)((B)D)D)B.$$

A proof of the rule is an object of the above type. The following gives such a proof.

$$(A)(B)(x)(y)y(B,x(B),(z)z):$$

$$(A:\mathrm{Set})(B:\mathrm{Set})((C:\mathrm{Set})(A)C)((D:\mathrm{Set})((A)D)((B)D)D)B.$$

We see now that the old rules of proof, in the sense of functions effecting an inference step, can be treated in precisely the same way as any other type-theoretical objects, assumed, applied and abstracted over. The form of a problem to prove suggests what forms of rules of proof to assume during its solution.

The constructive uniqueness axiom for lines and points and the uniqueness axiom for the orthogonal construction both have four disjuncts in the conclusion, and the uniqueness axiom for the parallel construction has three. Type disjunction cannot in general be iterated, but we shall instead formalize these axioms by what corresponds to three- and four-place type disjunctions (where a number indicates the number of arguments).

The different geometries are now written as contexts.

Apartness geometry

(Point : Set, Line : Set,

DiPt : (Point)(Point)Set, DiLn : (Line)(Line)Set, Con : (Line)(Line)Set,

irr_dipt : (a : Point)Non(DiPt(a, a)),

irr_diln : (l : Line)Non(DiLn(l, l)),

irr_con : (l : Line)Non(Con(l, l)),

apart_dipt : (a, b, c : Point)(DiPt(a, b))Vel(DiPt(a, c), DiPt(b, c)),

apart_diln : (l, m, n : Line)(DiLn(l, m))Vel(DiLn(l, n), DiLn(m, n)),

apart_con : (l, m, n : Line)(Con(l, m))Vel(Con(l, n), Con(m, n)),

ln : (a, b : Point)(DiPt(a, b))Line,

pt : (l, m : Line)(Con(l, m))Point,

Apt : (Point)(Line)Set,

inc_ln1 : (a, b : Point)(w : DiPt(a, b))Non(Apt(a, ln(a, b, w))),

inc_ln2 : (a, b : Point) (w : DiPt(a, b))Non(Apt(b, ln(a, b, w))),

inc_pt1 : (l, m : Line)(w : Con(l, m))Non(Apt(pt(l, m, w), l)),

inc_pt2 : (l, m : Line)(w : Con(l, m))Non(Apt(pt(l, m, w), m)),

el_ax : (a, b : Point)(l, m : Line)(DiPt(a, b))(DiLn(l, m))

Vel4(Apt(a, l), Apt(a, m), Apt(b, l), Apt(b, m)),

cmp_apt_dipt : (a, b : Point)(l : Line)(Apt(a, l))Vel(DiPt(a, b), Apt(b, l)),

cmp_apt_diln : (a : Point)(l, m : Line)(Apt(a, l))Vel(DiLn(l, m), Apt(a, m)),

cmp_con_diln : (l, m, n : Line)(Con(l, m))Vel(DiLn(m, n), Con(l, n)))

Projective geometry

Add the following to apartness geometry:

$$\text{proj\_ax} : (l, m : \text{Line})(\text{DiLn}(l, m))\text{Con}(l, m).$$

Affine geometry

Add the following to apartness geometry:

$$\text{par} : (\text{Line})(\text{Point})\text{Line},$$

$$\text{is\_par} : (l : \text{Line})(a : \text{Point})\text{Non}(\text{Con}(\text{par}(l, a), l)),$$

$$\text{inc\_par} : (a : \text{Point})(l : \text{Line})\text{Non}(\text{Apt}(a, \text{par}(l, a))),$$

$$\text{uni\_par} : (a : \text{Point})(l, m : \text{Line})(\text{DiLn}(l, m))\text{Vel3}(\text{Apt}(a, l), \text{Apt}(a, m), \text{Con}(l, m)).$$

Orthogonal geometry

Add the following to affine geometry:

$$\text{Unort} : (\text{Line})(\text{Line})\text{Set},$$

$$\text{cmp\_con\_unort} : (l, m : \text{Line})\text{Vel}(\text{Con}(l, m), \text{Unort}(l, m)),$$

$$\text{apart\_obl} : (l, m, n : \text{Line})(\text{Con}(l, m))\ (\text{Unort}(l, m))$$

$$((A : \text{Set})(((\text{Con}(l, n))(\text{Unort}(l, n)))A)((\text{Con}(m, n))(\text{Unort}(m, n)))A)A,$$

$$\text{ort} : (\text{Line})(\text{Point})\text{Line},$$

$$\text{is\_ort} : (l : \text{Line})(a : \text{Point})\text{Non}(\text{Unort}(\text{ort}(l, a), l)),$$

$$\text{inc\_ort} : (a : \text{Point})(l : \text{Line})\text{Non}(\text{Apt}(a, \text{ort}(l, a))),$$

$$\text{uni\_ort} : (a : \text{Point})(l, m, n : \text{Line})(\text{DiLn}(l, m))$$

$$\text{Vel4}(\text{Apt}(a, l), \text{Apt}(a, m), \text{Unort}(l, n), \text{Unort}(m, n)).$$

If we want to work with the classical axiomatization of a geometry, it is enough to make the hypothesis that we have a proof of the law of double negation at hand. In type-logical terms, we add to the appropriate context the following:

$$\text{dn} : ((A : \text{Set})(B : \text{Set})((C : \text{Set})(A)C)B)A.$$

The computability of classical solutions of problems would require such a hypothetical function dn to be computable.

Next we give the axioms in the notation of ALF. We shall not use the type-logical definitions, for the reason that ALF in its present form does not permit their effective use. Comments inside the code are written (* like this *).

```
Apartness_Geometry is

(* apartness axioms for DiPt, DiLn and Con *)
[ Point:Set;Line:Set;
  DiPt:(Point;Point)Set;
  DiLn:(Line;Line)Set;
  Con:(Line;Line)Set;
  irr_dipt:(a:Point;A:Set;DiPt(a,a))A;
  irr_diln:(l:Line;A:Set;DiLn(l,l))A;
  irr_con:(l:Line;A:Set;Con(l,l))A;
  apart_dipt:(a,b,c:Point;DiPt(a,b);
              A:Set;(DiPt(a,c))A;(DiPt(b,c))A)A;
  apart_diln:(l,m,n:Line;DiLn(l,m);
              A:Set;(DiLn(l,n))A;(DiLn(m,n))A)A;
  apart_con:(l,m,n:Line;Con(l,m);
             A:Set;(Con(l,n))A;(Con(m,n))A)A;

(* line and point constructions and their properties *)
  ln:(a,b:Point;DiPt(a,b))Line;
  pt:(l,m:Line;Con(l,m))Point;
  Apt:(Point;Line)Set;
  inc_ln1:(a,b:Point;w:DiPt(a,b);A:Set;Apt(a,ln(a,b,w)))A;
  inc_ln2:(a,b:Point;w:DiPt(a,b);A:Set;Apt(b,ln(a,b,w)))A;
  inc_pt1:(l,m:Line;w:Con(l,m);A:Set;Apt(pt(l,m,w),l))A;
  inc_pt2:(l,m:Line;w:Con(l,m);A:Set;Apt(pt(l,m,w),m))A;

(* uniqueness axiom *)

  el_ax:(a,b:Point;l,m:Line;DiPt(a,b);DiLn(l,m);
        A:Set;(Apt(a,l))A;(Apt(a,m))A;(Apt(b,l))A;(Apt(b,m))A)A;

(* compatibility axioms *)
  cmp_apt_dipt:(a,b:Point;l:Line;Apt(a,l);
                A:Set;(DiPt(a,b))A;(Apt(b,l))A)A;
  cmp_apt_diln:(a:Point;l,m:Line;Apt(a,l);
                A:Set;(DiLn(l,m))A;(Apt(a,m))A)A;
  cmp_con_diln:(l,m,n:Line;Con(l,m);
                A:Set;(DiLn(m,n))A;(Con(l,n))A)A ]

Projective_Geometry is Apartness_Geometry +

[ proj_ax:(l,m:Line;DiLn(l,m))Con(l,m) ]
```

```
Affine_Geometry is Apartness_Geometry +

(* the parallel construction, its properties and uniqueness *)
[ par:(Line;Point)Line;
  is_par:(l:Line;a:Point;A:Set;Con(par(l,a),l))A;
  inc_par:(a:Point;l:Line;A:Set;Apt(a,par(l,a)))A;
  uni_par:(a:Point;l,m:Line;DiLn(l,m);
           A:Set;(Apt(a,l))A;(Apt(a,m))A;(Con(l,m))A)A ]

Orthogonal_Geometry is Affine_Geometry +

(* axioms for Unort *)
[ Unort:(Line;Line)Set;
  cmp_con_unort:(l,m:Line;A:Set;(Con(l,m))A;(Unort(l,m))A)A;
  apart_obl:(l,m,n:Line;Con(l,m);Unort(l,m);
             A:Set;(Con(l,n);Unort(l,n))A;(Con(m,n);Unort(m,n))A)A;

(* the orthogonal construction, its properties and uniqueness *)
  ort:(Line;Point)Line;
  is_ort:(l:Line;a:Point;A:Set;Unort(ort(l,a),l))A;
  inc_ort:(a:Point;l:Line;A:Set;Apt(a,ort(l,a)))A;
  uni_ort:(a:Point;l,m,n:Line;DiLn(l,m);A:Set;(Apt(a,l))A;
           (Apt(a,m))A;(Unort(l,n))A;(Unort(m,n))A)A ]
```

A formalization with explicit logical operations is straightforward.

12. Historical remarks

Axiomatizations of synthetic geometry have had as their model the one given in Hilbert's *Grundlagen der Geometrie* at the turn of the century. Hilbert's axiomatization was classical, based on the (then brand new) idea of existence as consistency, instead of the older concept of existence as something constructed. Hilbert gave explicit axioms of incidence, but left the properties of point equality and line equality as part of unstated general principles.

When Heyting in the twenties tried to give a constructive axiomatization of geometry, he started from the apartness relations for points and for lines, relations whose negations were the equality relations used by Hilbert's classical axiomatization. Heyting gave the fundamental apartness axioms for distinct points and distinct lines. But instead of introducing the apartness of a point from a line, he followed Hilbert in using incidence as a basic relation. A point was defined to be apart from a line if it was distinct from every point incident with the line. Thus, if point a is apart from line l and point b incident with l, they are distinct points. This is in the present axiomatization an easy consequence of the compatibility axioms:

Theorem 12.1. $\mathrm{Apt}(a, l) \,\&\, \mathrm{Inc}(b, l) \rightarrow \mathrm{DiPt}(a, b)$.

Proof. From IV, $\mathrm{Apt}(a, l) \to \mathrm{DiPt}(a, b) \vee \mathrm{Apt}(b, l)$. So if $\mathrm{Apt}(a, l) \,\&\, \mathrm{Inc}(b, l)$, $\mathrm{DiPt}(a, b)$ remains. □

We get also a dual result,

Theorem 12.2. $\mathrm{Apt}(a, l) \,\&\, \mathrm{Inc}(a, m) \to \mathrm{DiLn}(l, m)$.

In 1959, Heyting defined parallelism of two distinct lines l, m by the condition $(\forall x : \mathrm{Pt})(\mathrm{Inc}(x, l) \to {\sim}\,\mathrm{Inc}(x, m))$, instead of introducing the constructively basic relation of line convergence used here. Thus, the idea of adding a construction postulate to incorporate parallelism did not arise in that context, but instead a constructively unexplained existence axiom was used [3, p. 163]. Heyting's definition of parallelism for distinct lines can be shown to follow from $\mathrm{Par}(l, m)$ in the present axiomatization:

Theorem 12.3. *Assume* $\mathrm{Par}(l, m)$ *and* $\mathrm{DiLn}(l, m)$. *Then* $\mathrm{Inc}(a, l) \to \mathrm{Apt}(a, m)$.

Proof. By the uniqueness axiom for parallels, $\mathrm{DiLn}(l, m) \to \mathrm{Apt}(a, l \mid m) \vee \mathrm{Con}(l, m)$. Since $\mathrm{Par}(l, m)$ and $\mathrm{Inc}(a, l)$ were assumed, only $\mathrm{Apt}(a, m)$ remains. □

Corollary 12.4. $\mathrm{DiLn}(l, m) \,\&\, \mathrm{Inc}(a, l \cdot m) \to \mathrm{Con}(l, m)$.

Proof. By using $\mathrm{DiLn}(l, m) \to \mathrm{Apt}(a, l \mid m) \vee \mathrm{Con}(l, m)$. □

Hilbert and Heyting both had in mind an axiomatization of geometry that would, as far as possible, characterize the geometry of the real plane and space, constructively for Heyting of course, and classically for Hilbert. With Heyting, this objective was tied to the attempt to reduce mathematics to the constructive arithmetic of the integers. (See also [11] for Heyting's geometrical work.)

We reason with ease from geometrical figures, perceiving as it were geometrical facts from them. But at the same time we feel that the figures are not the essential thing: It would not matter if they were badly drawn, say. There is instead something behind the figures that is responsible for the compelling experience we have. The present axiomatization started from admitting these facts. Constructive type theory, developed by Per Martin-Löf starting in 1970 [8], proved to be particularly suitable for representing geometric construction and inference. At the same time, it responded perfectly to the further aim of creating a constructive theory of geometric constructions. Martin-Löf [8] established the connection between type theory and computer programming. On its basis, the idea of viewing solutions of geometric problems as algorithms could be turned into reality. Our computer implementation in Section 11 uses the very efficient notation of higher-level type theory found by Martin-Löf in the latter 1980s.

In a first version of the axiomatization of apartness geometry, the rules of construction were accompanied by reverse rules that took out of a line the two distinct points

through which it was constructed, and similarly for points: With function constants p_1 and p_2 operating on the set Line, and l_1, l_2 operating on Point, one infers from l: Line that $p_1(l)$: Point, that $p_2(l)$: Point, and that $\mathrm{DiPt}(p_1(l), p_2(l))$. If a line is of the form given by the construction rule, the functions p_1 and p_2 compute as $p_1(\ln(a, b)) = a$: Point and $p_2(\ln(a, b)) = b$: Point. Analogous rules are given for the constants l_1 and l_2. We can now define

$$\mathrm{DiLn}(l, m) = \mathrm{Apt}(p_1(l) \mid p_2(l), m),$$

$$\mathrm{DiPt}(a, b) = \mathrm{Apt}(a, l_1(b) \mid l_2(b)).$$

The axioms of apartness geometry can be condensed into

1. $\sim \mathrm{Apt}(p_1(l), l)$, $\sim \mathrm{Apt}(p_2(l), l)$,
2. $\mathrm{Apt}(p_1(l) \mid p_2(l), m) \rightarrow \mathrm{Apt}(p_1(n) \mid p_2(n), l \mid m)$

and the dual forms of these for points, and the axiom stating that Con is an apartness relation. This really is a geometry of lines obtained as extensions of finite line segments, as in Euclid. The compatibilities of equality and apartness are derivable, as is Heyting's definition of apartness of a point from a line and the rest, but at the expense of very complicated proofs, with cases upon cases, accumulating one after the other. A further drawback of the axiomatization was that introducing new ways of constructing objects, such as parallel and orthogonal lines in addition to connecting lines, was not straightforward.

After seeing the above axiomatization of apartness geometry, Martin-Löf suggested simplifying it by including DiPt and DiLn and their axioms as basic, by adding the compatibility axioms that were derivable earlier, and by reformulating the crucial axiom 2 so that selectors p_1, p_2 are not needed. This framework is quite attractive in that it treats on a par geometric objects that are of a definite kind even though constructed in different ways. For example, the construction of parallel and orthogonal lines could be joined to the connecting lines of apartness geometry.

Proceeding on this line, I also arrived at the very simple formulation of axiom III for the apartness relation Apt. Its classical counterpart is

$$\mathrm{Inc}(a \cdot b, l \cdot m) \rightarrow \mathrm{EqPt}(a, b) \vee \mathrm{EqLn}(l, m).$$

I was recently surprised to discover this axiom in the paper Skolem [10]. There we also find the equivalence properties of point and line equalities as explicit axioms, as well as the classical counterparts to my two first compatibility axioms. Skolem's axiomatization is developed further in Ketonen's [5] application of Gentzen's deductive systems to the proof theory of geometry.

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References

[1] A. Heyting, Intuitionistische Axiomatiek der Projektieve Meetkunde (P. Noordhoff, Groningen, 1925).

[2] A. Heyting, Zur intuitionistischen Axiomatik der projektiven Geometrie, Math. Ann. 98 (1927) 491–538.

[3] A. Heyting, Axioms for intuitionistic plane affine geometry, in L. Henkin et al., eds., The Axiomatic Method (North-Holland, Amsterdam, 1959) 160–173.

[4] D. Hilbert, Grundlagen der Geometrie (Teubner, Leipzig, 1899).

[5] O. Ketonen, Untersuchungen zum Prädikatenkalkül, publ. as Annales Academiae Scientiarum Fennicae, Ser. A, No. 23, 1944.

[6] P. Mäenpää The art of analysis: Logic and history of problem solving, Doctoral dissertation, University of Helsinki, Faculty of Humanities (1993).

[7] L. Magnusson and B. Nordström, The ALF proof editor and its proof engine, in: H. Barendregt and T. Nipkow, eds., Types for Proofs and Programs, Lecture Notes in Computer Science 806 (Springer, Heidelberg, 1994) 213–237.

[8] P. Martin-Löf, Constructive mathematics and computer programming, in: L. Cohen et al., eds., Logic, Methodology and Philosophy of Science VI (North-Holland, Amsterdam, 1982) 153–175.

[9] A. Ranta, Type-Theoretical Grammar (Oxford University Press, Oxford, 1994).

[10] T. Skolem, Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen, as reprinted in: J. Fenstad, ed., Skolem's Selected Works in Logic (Universitetsforlaget, Oslo, 1970) 103–136.

[11] D. van Dalen, Heyting and intuitionistic geometry, in: P. Petkov, ed., Mathematical Logic (Plenum, New York, 1990) 19–27.