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# Baire generic histograms of wavelet coefficients and large deviation formalism in Besov and Sobolev spaces

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ABSTRACT

Histograms of wavelet coefficients are expressed in terms of the wavelet profile and the wavelet density. The large deviation multifractal formalism states that if a function $f$ has a minimal uniform Hölder regularity then its Hölder spectrum is equal to the wavelet density. The purpose of this paper is twofold. Firstly, we compute generically (in the sense of Baire's categories) these histograms in Besov $B_p^{s,q}(\mathbb{T})$ and $L^{p,s}(\mathbb{T})$ spaces, where $\mathbb{T}$ is the torus $\mathbb{R}^d/\mathbb{Z}^d$ (resp. in the Baire's vector space $V = \bigcap_{\varepsilon>0,\, p>0} B_p^{s(\frac{1}{p})-\frac{\varepsilon}{p},p}$ where $s : q \mapsto s(q)$ is a $C^1$ and concave function on $\mathbb{R}^+$ satisfying $0 \leqslant s' \leqslant d$ and $s(0) > 0$). Secondly, as an application, we deduce some extra generic properties for the histograms in these spaces, and study the generic validity of the large deviation multifractal formalism in Besov and $L^{p,s}$ spaces for $s > d/p$ (resp. in the above space $V$).



## 1. Introduction

Since we are interested in histograms of wavelet coefficients (resp. Hölder spectra), we can suppose from now on that the tempered distributions (resp. the locally bounded functions) we consider are periodic over $\mathbb{Z}^d$ (resp. periodic over $\mathbb{Z}^d$ and have vanishing integral on $[0,1]^d$). We take wavelets $\psi^{(i)}$, $i \in \{1,\dots,2^d-1\}$, in the Schwartz class as introduced in [8]. With the constant function 1, the periodized functions

$$2^{dj/2}\psi_{j,k}^{(i)}(x) := 2^{dj/2}\sum_{l\in\mathbb{Z}^d}\psi^{(i)}\big(2^j(x-l)-k\big),$$

$j \geqslant 0$, $k \in \{0,\dots,2^j-1\}^d$, $i \in \{1,\dots,2^d-1\}$, form an orthonormal basis of the space of functions in $L^2(\mathbb{T})$, where $\mathbb{T}$ is the torus $\mathbb{R}^d/\mathbb{Z}^d$. Since the $L^\infty$ normalization for wavelets is more convenient for our purpose, we denote

$$C_{j,k}^{(i)} = C_{j,k}^{(i)}(f) = 2^{dj}\int\limits_{[0,1]^d} f(t)\psi_{j,k}^{(i)}(t)\,dt$$

the wavelet coefficients of a function $f$ in $L^2(\mathbb{T})$ (with the usual modification when $f$ is a tempered distribution periodic over $\mathbb{Z}^d$).

Using the $L^\infty$ normalization, the Besov spaces $B_p^{s,q} := B_p^{s,q}(\mathbb{T})$ for ($s \in \mathbb{R}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$) are expressed by simple conditions (mixed $\ell^p$–$\ell^q$ norms) on wavelet coefficients (see [5,9]):

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$$f \in B_p^{s,q} \quad \Leftrightarrow \quad \left(\sum_j \left(\sum_{k,i} \left|C_{j,k}^{(i)} 2^{(s-\frac{d}{p})j}\right|^p\right)^{q/p}\right)^{1/q} < \infty \tag{1}$$

(with the usual modification when $p=\infty$ and/or $q=\infty$). This characterization does not depend on the choice of the above wavelets $\psi^{(i)}$.

This makes the use of Besov spaces very natural in the setting of signal and image processing. Signals and images are very often stored by their wavelet coefficients because of the fast decomposition algorithms and the sparsity of the representation; this sparsity allows, after quantization, to store the wavelet coefficients of large classes of signals and images in a very compressed form.

For $s>0$, $m \in \mathbb{N}$ and $m \leqslant s < m+1$, we recall that $B_2^{s,2}$ coincides with the usual Sobolev space $H^s$ and that if $s \notin \mathbb{N}$ then $B_\infty^{s,\infty} = C^s$ where $C^s$ is the Hölder space defined by $f \in C^s$ if there exist a constant $C>0$ and a polynomial $P$ of degree at most the integer part $[s] := m$ of $s$ such that

$$\forall x \ \forall x_0 \quad \left|f(x) - P(x-x_0)\right| \leqslant C|x-x_0|^s. \tag{2}$$

We also recall that

$$B_p^{s,q} \hookrightarrow C^{s-\frac{d}{p}} \quad \text{if } s > d/p,\ \ p>0 \text{ and } q>0.$$

When $p>1$, the $B_p^{s,q}$ are closely related to the usual Sobolev space $L^{p,s} = L^{p,s}(\mathbb{T}) := \{f \in L^p;\ (-\Delta)^{s/2} f \in L^p\}$ (remark that $L^{2,s} = H^s$), since (see [1,4])

$$\forall p>1\ \forall s>0 \quad B_p^{s,1} \hookrightarrow L^{p,s} \hookrightarrow B_p^{s,\infty} \tag{3}$$

and

$$\forall p>1\ \forall s>\varepsilon>0\ \forall q>0 \quad B_p^{s-\varepsilon,q} \hookrightarrow L^{p,s} \hookrightarrow B_p^{s+\varepsilon,q}. \tag{4}$$

Besov spaces for $p \geqslant 1$ (resp. $0<p<1$) and Sobolev spaces are Banach spaces (resp. quasi-Banach spaces) see [10], so that they are Baire spaces, i.e. any countable intersection of everywhere dense open sets is everywhere dense.

**Definition 1.** In a Baire space

- a countable intersection of (everywhere) dense open sets is called a (dense) $G_\delta$-set,
- if a property $\mathcal{P}$ holds on at least a (dense) $G_\delta$-set, we say that $\mathcal{P}$ holds generically,
- a result which holds generically is called generic.

The information concerning the Besov spaces that contain a tempered distribution $f$ can be stored through the knowledge of the scaling function which is defined for $p>0$ by

$$\eta_f(p) = \sup\left\{s \in \mathbb{R}:\ f \in B_p^{\frac{s}{p},p}\right\}. \tag{5}$$

This function was initially introduced in the context of fully developed turbulence, see [3] (where it is defined through $L^p$ norms of the increments of a locally bounded function $f$), and [4] (where the connection with Besov spaces is made). Thanks to the Besov embeddings $f$ belongs to the vector space

$$V = \bigcap_{\varepsilon>0,\ p>0} B_p^{(\eta(p)-\varepsilon)/p,p}, \tag{6}$$

where $\eta(p) = \eta_f(p)$.

In view of (1), the knowledge of the Besov spaces to which a tempered distribution $f$ belongs can be deduced from the histograms of wavelet coefficients of $f$ at all scales. Let us summarize what this is about; let $\alpha \in \mathbb{R}$, for each $j \geqslant 0$ let

$$N_j(\alpha) = \operatorname{Card}\left\{(k,i) \in \left\{0,\dots,2^j-1\right\}^d \times \left\{1,\dots,2^d-1\right\};\ \left|C_{j,k}^{(i)}\right| \geqslant 2^{-\alpha j}\right\}. \tag{7}$$

The wavelet profile $\nu_f$ is defined by

$$\nu_f(\alpha) = \lim_{\varepsilon\to 0^+}\left(\limsup_{j\to\infty}\left(\frac{\log N_j(\alpha+\varepsilon)}{\log(2^j)}\right)\right).$$

The wavelet density $\rho_f$ is defined by

$$\rho_f(\alpha) = \inf_{\varepsilon>0}\left(\limsup_{j\to\infty}\left(\frac{\log(N_j(\alpha+\varepsilon) - N_j(\alpha-\varepsilon))}{\log(2^j)}\right)\right).$$

A heuristic interpretation is that at scale $j$ (when $j \to \infty$) there are about $2^{\nu_f(\alpha)j}$ (resp. $2^{\rho_f(\alpha)j}$) wavelet coefficients of size $|C^{(i)}_{j,k}|$ larger than $2^{-\alpha j}$ (resp. of order $2^{-\alpha j}$).

Our motivation for computing the histograms of wavelet coefficients is the study of the validity of a large deviation multifractal formalism (we will write L.D. formalism). Multifractal analysis is concerned with the study of the pointwise regularity of functions, measured by the Hölder exponent. Let $x_0 \in \mathbb{R}^d$, $\alpha > 0$, we recall that a locally bounded function $f : \mathbb{R}^d \to \mathbb{R}$ is $C^\alpha(x_0)$ if there exists a polynomial $P$ of degree at most the integer part $[\alpha]$ of $\alpha$ and a constant $C$ such that in a neighborhood of $x_0$ we have $|f(x) - P(x - x_0)| \leqslant C|x - x_0|^\alpha$. The Hölder exponent $\alpha_f(x_0)$ of $f$ at $x_0$ is the supremum of all values of $\alpha$ such that $f$ is $C^\alpha(x_0)$. The pointwise Hölder regularity is summed up by computing the Hölder spectrum $d_f$ which associates to each $\alpha$ the Hausdorff dimension $d_f(\alpha)$ of the set $\{x;\ \alpha_f(x) = \alpha\}$. The L.D. formalism (see [6]) is a conjecture which asserts that if $f$ has a minimal uniform Hölder regularity (i.e. $f \in C^\varepsilon$ for $\varepsilon > 0$) then $d_f = \rho_f$.

Fix a $C^1$ strongly admissible function $\eta$ (in the sense of Definition 2 of the next section) and consider the associated vector space $V$ defined in (6). In [5], it was proved that $V$ is still a Baire space though it is not a Banach space, and the generic spectrum $d_f$ in $V$ was computed. This computation was also done in a given Besov or Sobolev space for $s > d/p$, nevertheless, if $s < d/p$, it was shown that generically the functions are not locally bounded, so that their Hölder spectra are defined for no value of $\alpha$.

To study the generic validity of the L.D. formalism in $V$ (resp. Besov or Sobolev space for $s > d/p$), we will compute the wavelet density $\rho_f$ for the functions of the (dense) $G_\delta$-set of $V$ (resp. Besov or Sobolev space for $s > d/p$) built in [5] and then compare the result to the generic spectrum $d_f$. We will, in particular, show that on these $G_\delta$-sets, the wavelet profile $\nu_f$ is equal to the wavelet density $\rho_f$.

In Section 2 we give the properties of the wavelet density, wavelet profile and the scaling function. We recall some connections between them. We also recall the derivation of the L.D. formalism using heuristic arguments.

In Section 3 we will state Baire's type results for both wavelet density and wavelet profile, in a given Besov or Sobolev space (see Theorem 1 in which we do not require the assumption $s > d/p$), resp. in a space $V$ associated to a fixed $C^1$ strongly admissible function $\eta$ (see Theorem 2). We deduce some extra generic properties for $\rho_f$ and $\nu_f$. We also study the generic validity of the L.D. formalism in a Besov or Sobolev space for $s > d/p$ (resp. $V$).

The proofs of Theorems 1 and 2 are given in Sections 4 and 5.

## 2. Functions $\nu_f$, $\rho_f$ and $\eta_f$ and L.D. formalism

### 2.1. Properties of $\nu_f$ and $\rho_f$

The function $\nu_f$ is nondecreasing, right-continuous and takes its values in $\{-\infty\} \cup [0, d]$. Furthermore, if $f$ is a periodic distribution, it has finite order, therefore there exist $\alpha_{\min} \in \mathbb{R}$ and $C_1 > 0$ such that

$$\forall (j,k,i) \quad \left|C^{(i)}_{j,k}\right| \leqslant C_1 2^{-\alpha_{\min} j}. \tag{8}$$

It follows that, if $\alpha < \alpha_{\min}$, $\nu_f(\alpha) = -\infty$. Similarly, the function $\rho_f$ is upper-semi-continuous, takes values in $\{-\infty\} \cup [0, d]$, and if $\alpha < \alpha_{\min}$, $\rho_f(\alpha) = -\infty$. Furthermore, $\nu_f$ is the increasing hull of $\rho_f$, i.e.

$$\nu_f(\alpha) = \sup_{\alpha' \leqslant \alpha} \rho_f(\alpha'). \tag{9}$$

In particular

$$\nu_f \geqslant \rho_f. \tag{10}$$

We call domain of definition of the function $\rho_f$ (resp. $\nu_f$) the set of $\alpha$ such that $\rho_f(\alpha) \neq -\infty$ (resp. $\nu_f(\alpha) \neq -\infty$).

In [6], it was proved that we have not guarantee that analyzing $f$ with a given $\rho_f$ using a different wavelet basis, we would recover $\rho_f$. We can only be sure to recover its increasing hull $\nu_f$.

### 2.2. Some connections with the scaling function

The following lemma proved in [2] shows that the scaling function can be deduced from the histograms of the wavelet coefficients.

**Lemma 1.** *Any periodic tempered distribution $f$ satisfies*

$$\forall p > 0 \quad \eta_f(p) = \inf_{\alpha \geqslant \alpha_{\min}} \big(\alpha p - \rho_f(\alpha) + d\big) = \inf_{\alpha \geqslant \alpha_{\min}} \big(\alpha p - \nu_f(\alpha) + d\big) \tag{11}$$

*where $\alpha_{\min}$ is any value satisfying* (8).

It follows that the information given by the Besov spaces which contain $f$ only yields the concave hull $\eta_f$ of the function $\nu_f$. Thus $\eta_f$ is concave on $]0,\infty[$, and whenever $\nu_f$ is not concave, it contains strictly more information on $f$ than $\eta_f$. We also deduce that for any $\alpha \in \mathbb{R}$

$$\rho_f(\alpha) \leqslant \nu_f(\alpha) \leqslant \inf_{p>0}\big(\alpha p - \eta_f(p) + d\big). \tag{12}$$

In particular, if $f \in B_p^{s,q}$ with $p \neq \infty$ then for any $\alpha \in \mathbb{R}$

$$\rho_f(\alpha) \leqslant \nu_f(\alpha) \leqslant \alpha p - sp + d. \tag{13}$$

*2.3. The heuristic derivation of the L.D. formalism*

Let us now recall the heuristic argument from which the L.D. formalism was derived. Though this argument cannot be transformed into a correct mathematical proof, it shows at least why that formula can be expected to hold. If a function $f$ has a minimal uniform Hölder regularity, it is proved in [7] that the Hölder exponent of $f$ at every point $x_0$ is given by the formula

$$\alpha_f(x_0) = \liminf_{j\to\infty}\left(\inf_{k,i} \frac{\log(|C_{j,k}^{(i)}|)}{\log(2^{-j} + |k2^{-j} - x_0|)}\right). \tag{14}$$

Consider a dyadic cube $\lambda$ of size $2^{-j}$ which contains a Hölder exponent $\alpha$ at $x_0$. Then (14) implies that $|C_{j,k}^{(i)}|$ is of the order of magnitude of $2^{-j\alpha}$. So we expect to find about $2^{jd_f(\alpha)}$ such coefficients, but we know that there are about $2^{j\rho_f(\alpha)}$ of them, hence the formula $d_f = \rho_f$.

*2.4. The scaling function*

To introduce Definition 2, we need to recall some other properties. For $0 < p < \infty$, let $q = 1/p$ and $s_f(q) := q\eta_f(1/q) = \frac{\eta_f(p)}{p}$. Clearly

$$s_f(q) = \sup\big\{s \in \mathbb{R}:\ f \in B_p^{s,p}\big\}. \tag{15}$$

Using Besov embeddings, it was proved in [5] that $s_f$ is increasing and concave on $]0,\infty[$ and that its right and left derivatives belong to $L^\infty$ and satisfy

$$\forall q \in\, ]0,\infty[, \quad \big(s_f'\big)_r(q) \leqslant d \quad \text{and} \quad \big(s_f'\big)_l(q) \leqslant d. \tag{16}$$

Note that $s_f$ is differentiable almost everywhere (we write a-e) in $]0,\infty[$ since it is increasing.

For the study of the L.D. formalism, we will need furthermore that the functions $f$ we consider have some uniform regularity, i.e. that there exists $\gamma > 0$ such that $f \in C^\gamma(\mathbb{T})$. This condition can be written $s_f(0^+) > 0$.

For simplicity of notations, we will only consider $C^1$ functions $s$. Natural generalization can be directly obtained for increasing concave functions $s$ (since they are a-e differentiable and their a-e derivatives are a-e continuous).

**Definition 2.** A function $\eta : \mathbb{R}^+ \to \mathbb{R}$ is $C^1$ strongly admissible if the function $s$ defined by $s(q) = q\eta(1/q)$ for $q > 0$ and $s(0) := s(0^+) := \lim_{q\to 0^+} s(q) > 0$, is concave and continuously differentiable on $\mathbb{R}^+$, and satisfies $0 \leqslant s' \leqslant d$.

Clearly if $\eta$ is $C^1$ strongly admissible, it is concave. And since $s$ is concave with $0 \leqslant s' \leqslant d$, there exists a critical value $q_c$ of $q$, such that if $q < q_c$, $s(q) > dq$ and if $q > q_c$, $s(q) < dq$, except in the $d$-degenerate case where $s(q) = s_0 + dq$ in which case $q_c = \infty$.

In the following we will not consider neither this case, nor the $r$-degenerate case where $s'$ is constant $r$ with $0 \leqslant r < d$ (i.e. if $s(q) = rq + s_0$) in which $q_c = s_0/(d-r)$, because in these cases the problem is reduced to just one point, and does not yield us any further information.

For $h \geqslant 0$, set

$$\tilde{s}(h) = \sup_{q>0}\big(s(q) - hq\big). \tag{17}$$

Remark that in the above $r$-degenerate cases, $\tilde{s}(q) = s_0$ if $h \geqslant r$, and $\infty$ if $h < r$.

In Section 4, we will need the following lemma which follows from the definition of $\tilde{s}$ and the properties of $s$.

**Lemma 2.** *Let $\eta$ be $C^1$ strongly admissible. Suppose that $s'$ is no constant. Then*

- *If $s'(\infty) := \lim_{q\to\infty} s'(q)$ then $\tilde{s}(h) = \infty$ for $h < s'(\infty)$.*

- *$\tilde{s}$ is real-valued, continuous and convex in $]s'(\infty), s'(0)]$.*
- *If $h = s'(q_h)$ with $0 \leqslant q_h < \infty$ then $\tilde{s}(h) = s(q_h) - hq_h$.*
- *$\tilde{s}(h) = s(0)$ for $h \geqslant s'(0)$.*
- *If $q_{\max} := \sup\{q \geqslant 0;\ \tilde{s}(s'(q)) = s(0)\}$ and $h_{\min} := s'(q_{\max})$ then $h_{\min} = \inf\{h \leqslant s'(0);\ \tilde{s}(h) = s(0)\}$, $\tilde{s}$ is constant $s(0)$ on $[h_{\min}, \infty[$, and $\tilde{s}$ is strictly decreasing in $]s'(\infty), h_{\min}]$. In particular $\tilde{s}$ is a bijective map from $]s'(\infty), h_{\min}]$ to $[s(0), \lim_{h\to(s'(\infty))^+} \tilde{s}(h)[$ and $\lim_{h\to(s'(\infty))^+} \tilde{s}(h) = \lim_{q\to\infty}(s(q) - qs'(q))$.*
- *$\tilde{s}(d) = s(0)$.*
- *$\tilde{s}(0) = \sup_{q>0} s(q) \geqslant \lim_{h\to(s'(\infty))^+} \tilde{s}(h)$.*
- *If $s'(\infty) > 0$ then $\tilde{s}(0) = \infty$.*
- *If $s'(\infty) = 0$ then $\tilde{s}(0) = \tilde{s}(s'(\infty))$.*

Remark that in the above $r$-degenerate cases, $\tilde{s}(0) = \infty$ if $r > 0$, and $s_0$ if $r = 0$, $\lim_{h\to(s'(\infty))^+} \tilde{s}(h) = s_0$ and $\tilde{s}(d) = s_0$.

## 3. Baire's results and conclusions

First, we state results holding generically for both wavelet density and wavelet profile in a Besov space and Sobolev space (resp. in $V$). Then we will give some generic conclusions for both them and the validity of the L.D. formalism.

**Theorem 1.** *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $s > 0$. Then generically, the domain of definition of the wavelet density in $B_p^{s,q}(\mathbb{T})$ is $[s - \frac{d}{p}, s]$ and*

$$\forall \alpha \in \left[s - \frac{d}{p}, s\right] \quad \nu_f(\alpha) = \rho_f(\alpha) = \alpha p - sp + d \tag{18}$$

(*with the convention $\frac{d}{p} = 0$ if $p = \infty$ and $\alpha p - sp + d = d$ if $\alpha = s$ and $p = \infty$*).

*If $p > 1$ and $s > 0$, the same result holds for $L^{p,s}(\mathbb{T})$.*

Note that unlike [5] we do not require the assumption $s > d/p$ in the above theorem. We also conclude that generically in $B_p^{s,q}$ and $L^{p,s}$, the wavelet profile $\nu_f$ is affine if $p \neq \infty$ and does not contain any strictly more information on $f$ than $\eta_f$ and that the wavelet density $\rho_f$ is nondecreasing and affine if $p \neq \infty$, and does not depend on the wavelet basis chosen in the Schwartz class.

From now on, the notation $[a, b)$ means the interval $[a, b]$ if $b < \infty$, resp. $[a, b[$ if $b = \infty$.

**Theorem 2.** *Let $\eta$ be a $C^1$ strongly admissible function. Let $V$ be the function space defined by (6). Then generically, the domain of definition of the wavelet density in $V$ is $[s(0), \lim_{q\to\infty}(s(q) - qs'(q)))$, and on this interval*

$$\nu_f(\alpha) = \rho_f(\alpha) = \inf_{p>0}\bigl(\alpha p - \eta(p) + d\bigr).$$

We conclude that generically in $V$, the wavelet profile $\nu_f$ is concave and does not contain any strictly more information on $f$ than $\eta_f$ and that the wavelet density $\rho_f$ is nondecreasing, concave, and does not depend on the wavelet basis chosen in the Schwartz class.

We are now ready to study the generic validity of the L.D. formalism. In [5], it was proved that for $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $s > d/p$, the domain of definition of the generic Hölder spectrum in $B_p^{s,q}$ (resp. $L^{p,s}$ if $p > 1$) is the interval $[s - \frac{d}{p}, s]$ and on this interval $d_f(\alpha) = \alpha p - sp + d$. Therefore, since our computations are done on the same $G_\delta$-sets, it follows from Theorem 1 that the L.D. formalism holds generically in $B_p^{s,q}$ (resp. $L^{p,s}$ if $p > 1$) for $s > d/p$. On the other hand, since $\eta$ is strongly admissible, then [5] implies that, in $V$, the domain of definition of the generic Hölder spectrum is the interval $[s(0), dq_c]$, and on this interval $d_f(\alpha) = \inf_{p\geqslant p_c}(\alpha p - \eta(p) + d)$ (where $p_c = 1/q_c$). The generic spectrum is composed of two parts: a part defined by $\alpha < \eta'(p_c)$ where the infimum is attained for $p > p_c$, and so $d_f(\alpha) = \inf_{p>0}(\alpha p - \eta(p) + d)$, and a part defined by $\eta'(p_c) \leqslant \alpha \leqslant d/p_c$ where the infimum is attained for $p = p_c$, and the spectrum is a straight line $d_f(\alpha) = \alpha p_c$. In our work we consider the same $G_\delta$-set and $s(0) \leqslant \eta'(p_c) = \tilde{s}(s'(q_c)) = s(q_c) - q_c s'(q_c) \leqslant dq_c$ and $\eta'(p_c) \leqslant \lim_{q\to\infty}(s(q) - qs'(q))$. (Remark that since $s$ is differentiable then for $q > q_c$ there exists $Q \in ]q_c, q[$ such that $(s(q) - s(q_c)) - qs'(q) = s'(Q)(q - q_c) - qs'(q) \geqslant s'(q)(q - q_c) - qs'(q) = -q_c s'(q)$. So if $s'(\infty) = 0$ then $\lim_{q\to\infty}(s(q) - qs'(q)) \geqslant dq_c$.) At point $\alpha = \eta'(p_c)$ the slope of the tangent to the concave function $\rho_f$ is $p_c$. Therefore thanks to Theorem 2 we conclude that the L.D. formalism holds (resp. may fail) generically in $V$ for $\alpha \in [s(0), \eta'(p_c)]$ (resp. $\alpha \in [\eta'(p_c), \lim_{q\to\infty}(s(q) - qs'(q)))$).

## 4. Proof of Theorem 1

We distinguish three cases: $p$ and $q$ finite, then $p = \infty$ and/or $q = \infty$, and the $L^{p,s}$ case.

*4.1. The case $0 < p < \infty$ and $0 < q < \infty$*

Let $a \geqslant 1$. We slightly modify the definition (7) of $N_j(\alpha)$ by setting for each $j \geqslant 1$

$$N_{j,a}(\alpha) = \mathrm{Card}\left\{(k,i);\ \left|C_{j,k}^{(i)}\right| \geqslant \frac{1}{j^a} 2^{-\alpha j}\right\} \tag{19}$$

and

$$\rho_f(\alpha, \varepsilon) = \limsup_{j\to\infty} \frac{\log(N_{j,a}(\alpha+\varepsilon) - N_{j,a}(\alpha-\varepsilon))}{\log(2^j)}. \tag{20}$$

**Lemma 3.**

$$\rho_f(\alpha) = \inf_{\varepsilon>0} \rho_f(\alpha, \varepsilon). \tag{21}$$

This result follows from the fact that since $\lim \frac{\log j}{j} = 0$ then for $j$ large enough

$$N_j\left(\alpha + \frac{\varepsilon}{2}\right) - N_j\left(\alpha - \frac{\varepsilon}{2}\right) \leqslant N_{j,a}(\alpha+\varepsilon) - N_{j,a}(\alpha-\varepsilon) \leqslant N_j(\alpha+2\varepsilon) - N_j(\alpha-2\varepsilon).$$

We first compute both the wavelet density and wavelet profile for a specific "saturating wavelet series" $F$ and then we will consider a dense $G_\delta$-set.

*4.1.1. Saturating wavelet series*

We fix a Besov space $B_p^{s,q}(\mathbb{T})$ where $s > 0$, $0 < p < \infty$ and $0 < q < \infty$. Let $j \geqslant 1$ and $k \in \{0, \ldots, 2^j - 1\}^d$ be given. Consider the irreducible representation

$$\frac{k}{2^j} = \frac{K}{2^J} \quad \text{where } K \in \mathbb{Z}^d - (2\mathbb{Z})^d. \tag{22}$$

Let

$$a = \frac{2}{p} + \frac{2}{q} + 1 \tag{23}$$

and

$$F = \sum_{j\geqslant 0,k,i} C_{j,k}^{(i)} \psi_{j,k}^{(i)} \tag{24}$$

where

$$C_{j,k}^{(i)} = \frac{1}{j^a} 2^{(\frac{d}{p}-s)j} 2^{-\frac{d}{p}J}. \tag{25}$$

In [5], it is proved that $F$ belongs to $B_p^{s,q}(\mathbb{T})$.

A straightforward computation yields the following lemma.

**Lemma 4.** *For each $1 \leqslant J \leqslant j$ there are $2^{dJ} - (\frac{2^J}{2})^d = C_d 2^{dJ}$ values of k satisfying* (22)*, where $C_d = (1 - 2^{-d})$.*

**Proposition 1.** *The wavelet density $\rho_F$ of the wavelet series F given by* (24) *and* (25) *is defined in $[s - \frac{d}{p}, s]$, and on this interval*

$$\nu_F(\alpha) = \rho_F(\alpha) = \alpha p - sp + d.$$

**Proof of Proposition 1.** Since

$$\forall (j,k,i) \quad \frac{1}{j^a} 2^{-sj} \leqslant \left|C_{j,k}^{(i)}\right| \leqslant \frac{1}{j^a} 2^{(\frac{d}{p}-s)j} \tag{26}$$

and both the left and right terms are attained then the wavelet density $\rho_F$ is defined in the interval $[s - \frac{d}{p}, s]$.

Let $\alpha \in [s - \frac{d}{p}, s]$ and $\varepsilon > 0$ be fixed. Let $j \geqslant 1$. We will compute the number $N_{j,a}(\alpha+\varepsilon) - N_{j,a}(\alpha-\varepsilon)$. The relation

$$\frac{1}{j^a} 2^{-j(\alpha+\varepsilon)} \leqslant \left|C_{j,k}^{(i)}\right| < \frac{1}{j^a} 2^{-j(\alpha-\varepsilon)} \tag{27}$$

is equivalent to $\alpha - \varepsilon < s + \frac{d}{p}\frac{J}{j} - \frac{d}{p} \leqslant \alpha + \varepsilon$ and $J \leqslant j$. This means that

$$j\left(\alpha - \varepsilon + \frac{d}{p} - s\right)\frac{p}{d} < J \leqslant j\left(\alpha + \varepsilon + \frac{d}{p} - s\right)\frac{p}{d} \quad \text{and} \quad J \leqslant j. \tag{28}$$

We denote $E_{j,\varepsilon}$ this set of $J$'s. We will first take $\alpha \in [s - \frac{d}{p}, s[$ and then $\alpha = s$.

- If $\alpha \in [s - \frac{d}{p}, s[$ then for $\varepsilon$ small enough

$$\left(\alpha + \varepsilon + \frac{d}{p} - s\right)\frac{p}{d} \leqslant 1. \tag{29}$$

  Lemma 4 yields

$$N_{j,a}(\alpha + \varepsilon) - N_{j,a}(\alpha - \varepsilon) = C_d \sum_{J \in E_{j,\varepsilon}} 2^{dJ}.$$

  It follows from (28) and (29) that there exists a constant $C$ such that

$$\frac{1}{C} 2^{pj(\alpha+\varepsilon+\frac{d}{p}-s)} \leqslant N_{j,a}(\alpha + \varepsilon) - N_{j,a}(\alpha - \varepsilon) \leqslant C 2^{pj(\alpha+\varepsilon+\frac{d}{p}-s)}.$$

  Consequently $\rho_F(\alpha, \varepsilon) = p(\alpha + \varepsilon + \frac{d}{p} - s)$ and $\rho_F(\alpha) = \alpha p - sp + d$.
- If $\alpha = s$ then thanks to (26) relation (27) is equivalent to

$$\frac{1}{j^a} 2^{-sj} \leqslant \left|C_{j,k}^{(i)}\right| < \frac{1}{j^a} 2^{-j(s-\varepsilon)}.$$

  So $j - \varepsilon \frac{p}{d} j < J \leqslant j$ and there exists a constant $C$ such that

$$\frac{1}{C} 2^{jd} \leqslant N_{j,a}(s + \varepsilon) - N_{j,a}(s - \varepsilon) \leqslant C 2^{jd}.$$

  Hence $\rho_F(s, \varepsilon) = d$ and $\rho_F(s) = d = sp - sp + d$. $\square$

*4.1.2. The dense $G_\delta$-set*

Since $s > 0$, $0 < p < \infty$ and $0 < q < \infty$ then $B_p^{s,q}$ is separable. We compute $\rho_f$ and $\nu_f$ in the $G_\delta$-set built in [5]. We slightly modify a dense sequence $(f_n)$ in $B_p^{s,q}$ by replacing for each $f_n$ the wavelet coefficients for $j \geqslant n$ by those of the saturating wavelet series $F$. We obtain a new dense sequence $(g_n)$. We set

$$A = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geqslant m} B(g_n, r_n) \tag{30}$$

where $B(g_n, r_n)$ denotes the open ball (using the norm (1)) of center $g_n$ and radius $r_n$. The set $A$ is clearly a countable intersection of dense open sets in $B_p^{s,q}$. So $A$ is a (dense) $G_\delta$-set in $B_p^{s,q}$. Take $r_n = \frac{1}{2n^a} 2^{-nd/p}$. Remark that

$$f \in A \quad \Leftrightarrow \quad \forall m\ \exists n = n_m \geqslant m; \quad \|f - g_n\|_{B_p^{s,q}} < r_n.$$

Hence

$$\forall (j,k,i) \quad \left|C_{j,k}^{(i)}(f) - C_{j,k}^{(i)}(g_{n_m})\right| 2^{(s-\frac{d}{p})j} < r_{n_m}.$$

But by construction, at scale $j = n_m$, $C_{j,k}^{(i)}(g_{n_m}) = C_{j,k}^{(i)}(F)$. It follows from both (25), (26), the choice of $r_n$ and the fact that $J \leqslant j$, that at scale $j = n_m$

$$\forall (k,i) \quad \left|C_{j,k}^{(i)}(f) - C_{j,k}^{(i)}(F)\right| < \frac{1}{2j^a} 2^{-sj}$$

and

$$\forall (k,i) \quad \frac{1}{2}\left|C_{j,k}^{(i)}(F)\right| \leqslant \left|C_{j,k}^{(i)}(f)\right| \leqslant \frac{3}{2}\left|C_{j,k}^{(i)}(F)\right|. \tag{31}$$

Using the fact that there is lim sup in (20), we deduce that $\rho_f \geqslant \rho_F$. Using Proposition 1 and property (13) we deduce Theorem 1.

*4.2. The case where p and/or $q=\infty$*

We only consider the case where $p=q=\infty$, i.e. the $C^s(\mathbb{T})$ case with $s>0$. Since is not separable, the argument in this case is slightly different from the previous one. The proof in the case where only one among $p$ and $q$ is equal to $\infty$ is similar and is therefore omitted.

For $n\in\mathbb{N}$ set

$$E_n=\left\{g\in C^s;\ \forall(j,k,i)\ \exists M\in\mathbb{Z}^*\ C_{j,k}^{(i)}(g)=M2^{-n}2^{-sj}\right\}.$$

**Lemma 5.** *For any $m\in\mathbb{N}$, the set $D_m:=\bigcup_{n\geqslant m}E_n$ is dense in $C^s$.*

**Proof of Lemma 5.** Let $s>0$ and $f\in C^s$, we know that its norm is $\|f\|=\sup_{(j,k,i)}|C_{j,k}^{(i)}2^{sj}|$. So there exists $C>0$ such that

$$\forall(j,k,i)\quad \left|C_{j,k}^{(i)}(f)\right|\leqslant C2^{-sj}. \tag{32}$$

Now fix $n\in\mathbb{N}$ and take $g$ with wavelet coefficients $C_{j,k}^{(i)}(g)$ equal to $2^{-n}2^{-sj}$ if $|C_{j,k}^{(i)}(f)|<2^{-n}2^{-sj}$, and $C_{j,k}^{(i)}(g)$ equals to $[C_{j,k}^{(i)}(f)2^n2^{sj}]2^{-n}2^{-sj}$ if $|C_{j,k}^{(i)}(f)|\geqslant 2^{-n}2^{-sj}$ (where the notation $[t]$ means always the integer part of $t$). Clearly $g\in E_n$ and $\|f-g\|<2^{-n}$. Since $2^{-n}$ tends to 0 then Lemma 5 holds.

Now define $A_n=E_n+B(0,\frac{1}{2}2^{-n})$, $\mathcal{A}_m=\bigcup_{n\geqslant m}A_n$ and $\mathcal{A}=\bigcap_m\mathcal{A}_m$. Lemma 5 and the fact that $D_m\subset\mathcal{A}_m$ imply that $\mathcal{A}$ is a countable intersection of open dense sets of $C^s$. Remark that

$$f\in\mathcal{A}\quad\Leftrightarrow\quad\forall m\ \exists n=n_m\geqslant m\ \exists g\in E_n;\ \|f-g\|<\frac{1}{2}2^{-n}.$$

Hence

$$\forall(j,k,i)\quad\left|C_{j,k}^{(i)}(f)-C_{j,k}^{(i)}(g)\right|<\frac{1}{2}2^{-n}2^{-sj}.$$

But from the definition of $E_n$

$$\forall(j,k,i)\quad\left|C_{j,k}^{(i)}(g)\right|\geqslant 2^{-n}2^{-sj}.$$

And this fact together with (32) imply that

$$\forall(j,k,i)\quad\frac{1}{2}2^{-n}2^{-sj}\leqslant\left|C_{j,k}^{(i)}(f)\right|\leqslant C2^{-sj}.$$

We deduce that $\rho_f(\alpha)=d$ if $\alpha=s$, and $-\infty$ else. Using property (13) we deduce Theorem 1. $\square$

*4.3. The Sobolev case*

Clearly the case $p=2$ was proved since $L^{2,s}=B_2^{s,2}$. The proof for $p\neq 2$ follows immediately from the embeddings (4) and the proof in the case $B_p^{s,q}$ with $0<q<\infty$ and is therefore omitted.

## 5. Proof of Theorem 2

We first study a saturating function adapted to $\eta$ and then we will consider a dense $G_\delta$-set.

*5.1. A saturating function adapted to $\eta(p)$*

As in the previous section we first consider a saturating function for $V$.

We set, with $J$ as defined in (22)

$$a(j,k)=\inf_{p>0}\left(\frac{d(j-J)-\eta(p)j}{p}\right). \tag{33}$$

The saturating function $F$ is given by the wavelet coefficients

$$C_{j,k}^{(i)}=\frac{2^{a(j,k)}}{j^{\log j}} \tag{34}$$

(with the convention $2^{-\infty}=0$).

As in [5], we can prove that $F$ belongs to the space $V$ defined in (6), and that

$$\forall p>0\quad\eta_F(p)=\eta(p). \tag{35}$$

**Proposition 2.** *The domain of definition of the wavelet density of $F$ is $[s(0), \lim_{q\to\infty}(s(q)-qs'(q)))$, and on this interval*

$$\nu_F(\alpha)=\rho_F(\alpha)=\inf_{p>0}\big(\alpha p-\eta(p)+d\big).$$

**Proof of Proposition 2.** We have

$$\sup_{p>0}\left(\frac{\eta(p)}{p}-\frac{d}{p}\left(1-\frac{J}{j}\right)\right)=\sup_{q>0}\left(s(q)-dq\left(1-\frac{J}{j}\right)\right)=\tilde{s}(h_{j,J})$$

where $\tilde{s}$ is the function given in (17), and $h_{j,J}=d(1-\frac{J}{j})$. The $h_{j,J}$ take discrete values between 0 and $d$ with spacing $d/j$. The $\tilde{s}(h_{j,J})$ take discrete values between $\tilde{s}(d)=s(0)$ and $\tilde{s}(0)$. It follows from Lemma 2 that the domain of definition of the wavelet density of $F$ is $[s(0), \lim_{q\to\infty}(s(q)-qs'(q)))$.

Lemma 3 holds also if we replace $a$ by $a_j := \log j$ because $\lim \frac{\log(j\log j)}{j}=0$. So, we will compute the number $N_{j,a_j}(\alpha+\varepsilon)-N_{j,a_j}(\alpha-\varepsilon)$ for $\alpha\in[s(0), \lim_{q\to\infty}(s(q)-qs'(q)))$.

- If $\alpha\in\,]s(0), \lim_{q\to\infty}(s(q)-qs'(q))[$ then for $\varepsilon>0$ small enough $]\alpha-\varepsilon,\alpha+\varepsilon[\,\subset\,]s(0), \lim_{q\to\infty}s(q)-qs'(q)[$. Thus relation

$$\frac{1}{j^{a_j}}2^{-j(\alpha+\varepsilon)}\leqslant\big|C_{j,k}^{(i)}\big|<\frac{1}{j^{a_j}}2^{-j(\alpha-\varepsilon)}. \tag{36}$$

  is equivalent to $\alpha-\varepsilon<\tilde{s}(h_{j,J})\leqslant\alpha+\varepsilon$ and $J\leqslant j$.
  Using Lemma 2, i.e. the fact that $\tilde{s}$ is a bijective map from $]s'(\infty),h_{\min}]$ to $[s(0), \lim_{q\to\infty}(s(q)-qs'(q))[$, this means that $\tilde{s}^{-1}(\alpha+\varepsilon)\leqslant h_{j,J}<\tilde{s}^{-1}(\alpha-\varepsilon)$ and $J\leqslant j$. Therefore

$$j\left(\frac{d-\tilde{s}^{-1}(\alpha-\varepsilon)}{d}\right)<J\leqslant j\left(\frac{d-\tilde{s}^{-1}(\alpha+\varepsilon)}{d}\right). \tag{37}$$

  It follows from Lemma 4 that there exists a constant $C$ such that

$$\frac{1}{C}2^{j(d-\tilde{s}^{-1}(\alpha+\varepsilon))}\leqslant N_{j,a_j}(\alpha+\varepsilon)-N_{j,a_j}(\alpha-\varepsilon)\leqslant C2^{j(d-\tilde{s}^{-1}(\alpha+\varepsilon))}.$$

  Consequently

$$\rho_F(\alpha,\varepsilon)=d-\tilde{s}^{-1}(\alpha+\varepsilon)\quad\text{and}\quad\rho_F(\alpha)=d-\sup_{\varepsilon>0}\big(\tilde{s}^{-1}(\alpha+\varepsilon)\big)=d-\tilde{s}^{-1}(\alpha).$$

  But from Lemma 2, $\tilde{s}^{-1}(\alpha)\in\,]s'(\infty),h_{\min}[$, $h_{\min}\leqslant s'(0)$, and $\sup_{q>0}(s(q)-\tilde{s}^{-1}(\alpha)q)=\alpha$. Thus since $s'$ is continuous then it follows from the mean value theorem that there exists a finite non-zero value $q_\alpha$ of $q$ (in fact $q_\alpha>q_{\max}$) for which the above supremum is attained, i.e. $\alpha=s(q_\alpha)-\tilde{s}^{-1}(\alpha)q_\alpha$ and $\tilde{s}^{-1}(\alpha)=s'(q_\alpha)$. So

$$\tilde{s}^{-1}(\alpha)=\frac{s(q_\alpha)}{q_\alpha}-\frac{\alpha}{q_\alpha}=\eta(p_\alpha)-\alpha p_\alpha.$$

  Hence

$$\rho_F(\alpha)=d-\tilde{s}^{-1}(\alpha)\geqslant\inf_{p>0}\big(\alpha p-\eta(p)+d\big).$$

  Using property (12) we deduce that

$$\rho_F(\alpha)=\inf_{p>0}\big(\alpha p-\eta(p)+d\big).$$

- If $\alpha=s(0)$ (resp. $\alpha=\lim_{q\to\infty}(s(q)-qs'(q))<\infty$) then for $\varepsilon>0$ small enough $]\alpha,\alpha+\varepsilon[\,\subset\,]s(0),\lim_{q\to\infty}(s(q)-qs'(q))[$ (resp. $]\alpha-\varepsilon,\alpha[\,\subset\,]s(0),\lim_{q\to\infty}(s(q)-qs'(q))[$). So relation (36) is equivalent to $\alpha<\tilde{s}(h_{j,J})\leqslant\alpha+\varepsilon$ and $J\leqslant j$. (resp. $\alpha-\varepsilon<\tilde{s}(h_{j,J})\leqslant\alpha$ and $J\leqslant j$). Therefore as above we obtain $\rho_F(\alpha)=d-h_{\min}=d-s'(q_{\max})$ where $q_{\max}$ was defined in Lemma 2 (resp. $\rho_F(\alpha)=d-(s'(\infty))^+$). And as above we obtain the desired result. □

*5.2. The dense $G_\delta$-set*

Clearly, the separability of the space $V$ follows from the fact that finite linear combinations of wavelets with rational coefficients are dense. As in Section 4, we set $A=\bigcap_{m\in\mathbb{N}}\bigcup_{n\geqslant m}B(g_n,r_n)$, where $r_n=2^{-n\log n}$ and $g_n$ a dense sequence in $V$ such that if $j\geqslant n$ the wavelet coefficients of $g_n$ are the same as those of the saturating function.

$A$ is clearly a countable intersection of dense open sets. In [5] it was proved that $r_n$ is chosen so that, at the scale $j=n$, the wavelet coefficients of a function $f$ in $B(g_n,r_n)$ differ from those of $F$ by (much) less than $\frac{1}{2}.2^{-n\log n}$. As in Section 4, using the fact that there is lim sup in (20), we deduce that $\rho_f\geqslant\rho_F$. Using Proposition 2 and property (12) we deduce Theorem 2.

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